TITLE: A THEORETICAL STUDY OF THE ELECTROSTATIC COALESCENCE OF WATER-IN-OIL EMULSIONS

FINAL REPORT

BY

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EXECUTIVE SUMMARY

The electrostatic interaction of two charged water-in-oil (W/O) emulsion droplets at arbitrary separation and under the influence of an applied uniform electric field is investigated theoretically. The mathematical investigation of this interaction is greatly simplified if the droplets are assumed to be perfectly conducting spheres. This is an adequate model of the two droplets because the dielectric constant of the water is so much greater than that of the sourrounding oil. The ratio of the droplet sizes is arbitrary and the line joining the droplet centres can be at any angle with respect to the direction of the applied electric field.

The potential distribution in the oil medium surrounding the two spheres is first determined in terms of bipolar coordinates. The electrostatic interaction of the two spheres is then obtained in two stages. In the presence of an electric field, two uncharged conducting spheres in the oil medium interact with one another and the interaction can be found by adapting a simple formula for the energy of an electric dipole in a uniform electric field. The additional energy of interaction due to charges on the droplet are determined by means of a standard charging process. This approach is simpler than an alternative force method, which involves integrating over the surface of one of the spheres.

In the course of the investigation, it is demonstrated that the two spheres cannot be at the same potential (for example grounded) simultaneously when the separation is large. For meaningful values of the charge on the spheres, the presence of the electric field must result in a potential drop between the two spheres when widely separated. The interaction energy of the two spheres in terms of bipolar coordinates can be

transformed into a series in inverse distances between the spheres. In the case of uncharged spheres at large separations, the leading term in this series represents two electric dipoles pointing in the same direction and therefore attracting one another, a well-known result. Also the corresponding series for that part of the interaction energy due to the charges on the spheres is consistent with a result obtained by a twin multipole expansion. When the two spheres are almost touching, then the use of bipolar coordinates leads to slowly convergent series for the interaction energy. The force between two perfectly conducting spheres in contact under the influence of an applied electric field can be calculated by using so-called tangent-spheres curvilinear orthogonal coordinates. The special case of two equal spheres with the line of centres parallel to the applied electric field is examined.

Finally, the distortion of a spherical W/O droplet to a spheroid on application of an electric field is determined. The eccentricity of the spheroid is expressed in terms of the applied field, the radius of the original spherical shape, the oil/water interfacial tension, and the charge on the droplet. A small departure from a spherical shape is indicated for a typical electric field of 1 kV/cm.

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I. INTRODUCTION

This report is mainly concerned with the theoretical treatment of the electrostatic interaction of two charged water-in-oil (henceforth W/O) droplets under an applied uniform electric field. We begin with a brief explanation of the motivation for considering this problem and also with a summary of the relevant background. The co-production of water with oil in enhanced oil recovery or in thermal in-situ recovery of heavy oil and bitumen can create W/O emulsion droplets. These are an undesirable by-product since they give rise to problems of transport, corrosion and removal of water from the oil. W/O emulsions are not easily broken chemically. The problem is getting the chemical to reach the oil/water interface through the continuous oil phase. On the other hand, since oils are electrically non-conductive, the application of high voltage a.c or (less commonly) d.c. or pulsating electric fields is found to promote coalescence of W/O emulsions.

In the presence of an electric field, W/O droplets acquire electric dipole moments pointing in the same direction and hence attract one another. In the case of an a.c. field, the relaxation time of the droplet is usually much less than the period of the electric field. Because of the dipole-dipole attraction, chain formation is observed with both a.c. and d.c. high voltages [1-3]. These chains occur in the general direction of maximum field strength for non-uniform electric fields. Chain formation followed by coalescence is one of the mechanisms proposed for separation of liquid-liquid dispersions in an electric field. However, the chain formation can be undesirable if the conducting structure of chains occupies the whole emulsion, so that the electrodes are shorted out. An important problem for the future, therefore, is to understand the mechanisms governing the

formation of these chains, with a view to determining the optimum conditions for coalescence.

It should be mentioned that the phenomenon of chain formation also occurs with ferrofluids [4]. Ferromagnetic colloidal particles, coated with surfactant for stability in a non-magnetic fluid, are aligned in a suspension by a magnetic field and in a strong field chains of particles are observed. There are differences between W/O droplets in an electric field and ferromagnetic particles in a ferrofluid. The particles in ferrofluids have permanent magnetic dipoles in a saturation state. The emulsion droplets are much larger than the ferromagnetic particles and electric dipoles are induced only when the electric field is applied. Monte Carlo calculations on two-dimensional systems of spherical particles with permanent magnetic dipoles suspended in a non-magnetic fluid and in the presence of a strong magnetic field have been performed to predict their chain formation [5,6]. It may be worthwhile to apply the theoretical methods of studying ferrofluids to W/O emulsions under the influence of a strong electric field.

The determination of the electrostatic forces acting between colliding W/O emulsion droplets in the presence of an electric field is an essential first step in formulating the kinetics of coalescence. Two significant theoretical differences emerge between electrostatic coalescence of W/O emulsions and coalescence by non-electrical means of oil/water emulsions in general. The collision efficiency is defined as the fraction of collisions between two approaching droplets which leads to coalescence. In electrostatic coalescence the efficiency depends on the angle between the line joining the centres of the two colliding droplets and the direction of the applied field, whether a.c. or d.c. The other difference relates to

a.c. fields, when the coalescence frequency factor depends on time. The angle and time dependences are complications which make the calculation of coalescence rates more difficult. In an earlier report [7], this writer discussed the so-called population-balance equation for coalescence in an electric field where an attempt was made to include these additional effects.

The mathematical derivation of the electrostatic interaction of two emulsion droplets under the influence of an applied electric field is greatly simplified if the two droplets are assumed to be perfectly conducting spheres. Such a model should be a good approximation because the dielectric constants of the water and oil are in the ratio of about 80 to 2 or 3. Three methods of determining the interaction of two charged conducting spheres at arbitrary separation in a uniform electric field have been investigated in the literature. These are (i) successive imaging or reflections, (ii) twin multipole expansions, and (iii) use of bipolar coordinates. All these methods are inconvenient at very small separations between the spheres. The imaging method has been used by Krasny-Ergen [8,9] and by Smith and Rungis [10]. The method of multipole expansions was described in some detail in the aforementioned report by the writer (see also Ross [11]).

Consider two equal perfectly conducting spherical droplets of radius a, at separation R between their centres and carrying equal charges Q. The line of centres is assumed parallel to the electric field E and ϵ_o is the dielectric constant of the oil in which the droplets are immersed. Then the twin multipole expansion leads to the following interaction energy

$$V(R) = -2\epsilon_o E^2 \cdot \frac{a^6}{R^3}\left[1 + \frac{2}{3}\left(\frac{a}{R}\right)^3 + \ldots\right] + \frac{Q^2}{\epsilon_o R}\left[1 - \left(\frac{a}{R}\right)^3 + \ldots\right] \qquad (1)$$

For uncharged droplets at large separation, a good approximation which is often quoted in the literature is [12]

$$V(R) \simeq -\frac{2\epsilon_o E^2 a^6}{R^3} \qquad (2)$$

This leading term is a poor approximation, however, when the two spheres are almost touching.

Bipolar coordinates have been used by Davis [13] and by Panchenkov and Tsabek [14,15]. A preliminary account was given by the writer in his previous report [7] (see also Feat and Levine [16]). Davis treated the general case of two unequal spheres, both in respect to radius and potential or charge, at arbitrary angle between the line joining the centres of the two spheres and the direction of the applied electric field. He determined the force between the two spheres (i.e. the derivative $-\partial V(R)/\partial R$). In this report we shall reproduce the basic expressions for the the potential distributions obtained by Davis, whose paper is greatly condensed. The force expressions derived by Davis are very complicated, and indeed he stated that space did not permit the explicit inclusion of all the details. However, it is much simpler to aim directly for the interaction energy by using two theorems from classical electrostatics. For the part of this energy which corresponds to the term proportional to Q^2 in equation (1) above, one uses a simple charging process. For the part which represents the interaction of two uncharged droplets, one uses the theorem that the

energy of an electric dipole vector $\underline{m}$ in an electric vector field $\underline{E}$ is simply $-\underline{m}\cdot\underline{E}/2$. The major part of this report is devoted to developing this approach.

Suitable coordinates for determining the force between two conducting spheres in contact in a uniform electric field are the so-called tangent-sphere system [17-20]. In Chapter V, we calculate the force between two equal spheres in contact with the line of centres parallel to the electric field. An integral formula is obtained for the force. The use of tangent-sphere coordinates has the advantage of providing an exact expression for the force between two perfectly conducting spheres in contact, under the influence of an applied electric field. O'Meara and Saville [21] have used tangent-sphere coordinates to calculate the force between two unequal touching spheres.

In most of the literature on emulsions, the droplets are assumed to be spherical. It is natural to inquire to what extent a W/O emulsion droplet is distorted on applying an electric field and this problem is treated in Chapter VI of this report. It is assumed that to a first approximation the droplet takes the shape of a prolate spheroid and two models are considered, a dielectric spheroid and a perfectly conducting spheroid. The eccentricity of the spheroid is expressed in terms of the applied field, the radius of the original spherical shape, the oil/water interfacial tension, and the charge on the droplet. In an electric field of 1 kV/cm, departure from a spherical shape is found to be small.

II. INTERACTION OF TWO CONDUCTING SPHERICAL DROPLETS IN A UNIFORM ELECTRIC FIELD: EQUAL SPHERES

We shall first obtain the potential distribution for two equal spheres in bipolar coordinates, reproducing in more detail the results given by Davis [13]. Since his paper is condensed, it seems advisable to develop the theory from the beginning. Two identical, perfectly conducting, spherical droplets of radius a are immersed in an oil medium of dielectric permeability ϵ with the line of centres O_1 and O_2 at an angle ψ with the direction of the applied electric field E; the distance $O_1O_2 = R$. Let x,y,z be the rectangular Cartesian co-ordinates with origin at the mid-point O of the line O_1O_2 which lies on the z-axis; the x- and y-axes are situated in the median plane through O normal to the z-axis. The direction of the electric field lies in the x-z plane.

Spherical Bipolar Coordinates

We introduce bipolar co-ordinates μ,η,ϕ [17,22]

$$x = \frac{c\sinh\eta\cos\phi}{(\cosh\mu-\cos\eta)}, \quad y = \frac{c\sinh\eta\sin\phi}{(\cosh\mu-\cos\eta)}, \quad z = \frac{c\sinh\mu}{(\cosh\mu-\cos\eta)} \tag{1}$$

The co-ordinate μ defines a sphere of radius $c/\sinh\mu$ and centre at $x = y = 0$, $z = c\cdot\coth\mu$. μ ranges from $-\infty$ to $+\infty$ and $\mu = 0$ is the median plane. The surfaces of the droplets O_1 and O_2 are defined by $\mu = \pm\mu_0$, $\mu_0 > 0$,

$$\frac{c}{a} = \sinh\mu_0, \quad \frac{R}{2a} = \cosh\mu_0 \tag{2}$$

The co-ordinate η defines a spindle-shaped surface formed by rotating about the z-axis, namely at $z = \pm c$ where $\mu = \pm\infty$. The range of η is $0 \leq \eta \leq \pi$, where on the z-axis $\eta = 0$ for $|z| > c$ and $\eta = \pi$ for $|z| < c$ (Figure 1). ϕ is the angle of rotation about the z-axis. Elements of length along the curvilinear μ, η, ϕ axes are $h_\mu d\mu$, $h_\eta d\eta$, $h_\phi d\phi$, where

$$h_\mu = h_\eta = \frac{c}{\cosh\mu - \cos\eta}, \quad h_\phi = \frac{c\sin\eta}{\cosh\mu - \cos\eta} \tag{3}$$

Solution of Laplace's Equation

The electrostatic potential in the continuous oil medium satisfies Laplace's equation which may be written as

$$\nabla^2 V_o = \frac{1}{h_\mu^3}\left[\frac{\partial}{\partial\mu}\left(h_\mu \frac{\partial V_o}{\partial\mu}\right) + \frac{1}{\sin\eta}\frac{\partial}{\partial\eta}\left(h_\mu \sin\eta \frac{\partial V_o}{\partial\eta}\right) + \frac{h_\mu}{\sin^2\eta}\frac{\partial^2 V_o}{\partial\phi^2}\right] = 0 \tag{4}$$

This equation only separates on setting $V_o = (\cosh\mu - \cos\eta)^{1/2} F$, whence equation (4) is transformed to

$$\frac{\partial^2 F}{\partial\mu^2} + \frac{1}{\sin\eta}\frac{\partial}{\partial\eta}\left(\sin\eta\frac{\partial F}{\partial\eta}\right) + \frac{1}{\sin^2\eta}\frac{\partial^2 F}{\partial\phi^2} - \frac{1}{4}F = 0 \tag{5}$$

This separates into $F = M(\mu)H(\eta)\Phi(\phi)$, where

$$\frac{d^2\Phi}{d\phi^2} = -m^2\Phi \tag{6}$$

$$\frac{d^2 M}{d\mu^2} = (n+\tfrac{1}{2})^2 M \tag{7}$$

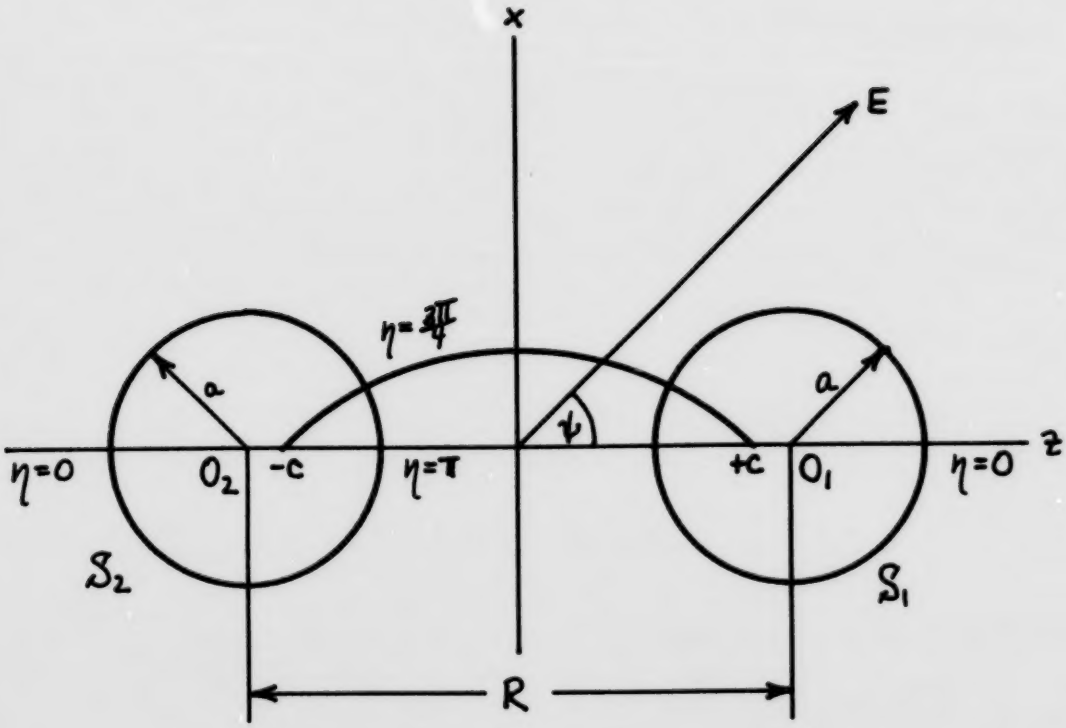

Figure 1. Bipolar coordinates for two identical spheres at separation R. The spheres S_1 and S_2 have coordinates $\mu = \mu_1$ and $\mu = -\mu_2$, respectively.

$$\frac{1}{\sin\eta}\frac{\partial}{\partial\eta}\left[\sin\eta\,\frac{dH}{\partial\eta}\right] - \frac{m^2H}{\sin^2\eta} = -n(n+1)H \tag{8}$$

For Φ to be continuous, m must be an integer or zero and in order for H to be finite at $\eta = 0$ and $\eta = \pi$, n must be an integer greater than or equal to m. Therefore, typical solutions of the original Laplace equation are

$$(\cosh\mu - \cos\eta)^{1/2} e^{\pm(n+\frac{1}{2})\mu} P_n^m(\cos\eta) \begin{matrix}\cos\\ \sin\end{matrix}(m\phi) \tag{9}$$

where $P_n^m(u) = (1-u^2)^{\frac{m}{2}} \dfrac{d^m P_n(u)}{du^m}$.

At large distances from the two spheres, the potential will have the form

$$V_o(\mu,\eta,\phi) = -Ez\cos\psi - Ex\sin\psi = -\frac{Ec\sinh\mu\cos\psi}{(\cosh\mu-\cos\eta)} - \frac{Ec\sin\eta\cos\phi\sin\psi}{(\cosh\mu-\cos\eta)} \tag{10}$$

This limits m to the values 0 and 1, so that we can write the solution of equation (4) in the form

$$V_o(\mu,\eta,\phi) = -\frac{Ec(\sinh\mu\cos\psi + \sin\eta\cos\phi\sin\psi)}{(\cosh\mu-\cos\eta)}$$

$$+ (\cosh\mu-\cos\eta)^{1/2}\left[\sum_{n=0}^{\infty}\left[A_n e^{(n+\frac{1}{2})\mu} + B_n e^{-(n+\frac{1}{2})\mu}\right]P_n(\cos\eta)\right.$$

$$\left. + \cos\phi \sum_{n=1}^{\infty}\left[C_n e^{(n+\frac{1}{2})\mu} + D_n e^{-(n+\frac{1}{2})\mu}\right]P_n^1(\cos\eta)\right] \tag{11}$$

The constants in equation (11) can be shown to be given by (Appendix A)

$$\begin{matrix} A_n \\ B_n \end{matrix} = \frac{1}{\sqrt{2}} e^{-(n+\frac{1}{2})\mu_0} \left[\frac{\Phi_0}{\cosh(n+\frac{1}{2})\mu_0} \pm \frac{2Ec(n+\frac{1}{2})\cos\psi}{\sinh(n+\frac{1}{2})\mu_0} \right] \tag{12}$$

$$C_n = D_n = \sqrt{2}Ec\sin\psi \frac{e^{-(n+\frac{1}{2})\mu_0}}{\cosh(n+\frac{1}{2})\mu_0} \tag{13}$$

where we assume the two spheres to have equal potentials Φ_0, yielding the simplest potential V_0. The more general coefficients obtained by Davis [13] reduce to those given in equations (12) and (13) for two identical spheres.

Force on a Sphere

The charge density on the surface S_2 of sphere O_2 ($\mu=-\mu_0$) is

$$\sigma_2 = -\frac{\epsilon}{4\pi}\frac{\partial V_o}{\partial n} = -\frac{\epsilon}{4\pi}\left[\frac{1}{h_\mu}\frac{\partial V_o}{\partial \mu}\right]_{\mu=-\mu_0} = -\frac{\epsilon}{4\pi}\frac{(\cosh\mu_0-\cos\eta)}{c}\left[\frac{\partial V_o}{\partial \mu}\right]_{\mu=-\mu_0} \tag{14}$$

noting that μ is increasing outwards from S_2. The force on O_2 in an arbitrary direction defined by the unit vector $\hat{r}$ is

$$\underset{\sim}{F} = \frac{2\pi}{\epsilon}\int_{S_2} \sigma_2^2 (\underset{\sim}{n}\cdot\hat{r})dS \tag{15}$$

where $\underset{\sim}{n}$ is the outward normal unit vector to S_2. Along the line of centres O_1 and O_2 in the z-direction

$$\underset{\sim}{n}\cdot\hat{r} = \underset{\sim}{n}\cdot\underset{\sim}{k} = -\left(\frac{1}{h_\mu}\frac{\partial z}{\partial \mu}\right)_{\mu=-\mu_0} = -\frac{\cosh\mu_0\cos\eta - 1}{(\cosh\mu_0 - \cos\eta)} \tag{16}$$

Noting that an element of area dS on S_2 is $h_\eta h_\phi d\eta d\phi$, the z-component of the force $\underset{\sim}{F}$ is

$$F_z = \frac{\epsilon}{4\pi}\int_0^{2\pi} d\phi \int_0^{\pi} d\eta \, \frac{\sin\eta(\cosh\mu_0\sin\eta - 1)}{(\cosh\mu_0 - \cos\eta)} \left[\frac{\partial V_o}{\partial \mu}\right]^2_{\mu=-\mu_0} \tag{17}$$

After evaluating the term $(\partial V_o/\partial\mu)$ which appears in the integral (see Appendix A), equation (17) can be written in the form

$$\begin{aligned} F_z = {} & \frac{\epsilon}{4}(Ec)^2\cos^2\psi \int_0^{\pi} J^2(\eta,-\mu_0) \, \frac{\sin\eta(\cosh\mu_0\sin\eta - 1)}{(\cosh\mu_0 - \cos\eta)} \, d\eta \\ & + \frac{\epsilon}{8}(Ec)^2\sin^2\psi \int_0^{\pi} K^2(\eta,-\mu_0) \, \frac{\sin\eta(\cosh\mu_0\sin\eta - 1)}{(\cosh\mu_0 - \cos\eta)} \, d\eta \\ & + \frac{\epsilon}{4}\Phi_0^2 \int_0^{\pi} L^2(\eta,-\mu_0) \, \frac{\sin\eta(\cosh\mu_0\sin\eta - 1)}{(\cosh\mu_0 - \cos\eta)} \, d\eta \\ & + 2Ec\Phi_0\cos\psi \int_0^{\pi} J(\eta,-\mu_0)L(\eta,-\mu_0) \, \frac{\sin\eta(\cosh\mu_0\sin\eta - 1)}{(\cosh\mu_0 - \cos\eta)} \, d\eta \end{aligned} \tag{18}$$

where the functions J, K, and L are defined by equations (A29-A31) in Appendix A.

From the analysis given by Davis [13], it would appear that the integrals in equation (18) can be evaluated by using known properties of Legendre polynomials. In order to obtain the interaction energy between the

two spheres, one needs to integrate F_z with respect to distance between spheres. Use can be made of the limited form of this interaction energy at large separations between the two spheres (i.e. the dipole-dipole interaction). Instead of embarking on this complicated task, particularly for unequal spheres, we consider an alternative and, indeed, simpler method of determining that part of the interaction energy which depends on the charge on the spheres - a method which is based on the familiar charging process.

Work of Charging Two Spheres of Equal Radius and Potential

We assume that the two droplets have the same radius a and, for simplicity, the same potential Φ_0. Their charges Q_1 and Q_2 will be different, because of the applied electric field. Integrating σ_2 over the surface S_2 of sphere O_2,

$$Q_2 = \int_{S_2} \sigma_2 dS_2 = \frac{-\epsilon}{4\pi} \int_0^{\pi} d\eta \int_0^{2\pi} d\phi \left[\frac{h_\eta h_\phi}{h_\mu} \frac{\partial V_o}{\partial \mu} \right]_{\mu=-\mu_0}$$

$$= \frac{-\epsilon c}{4\pi} \int_0^{\pi} d\eta \int_0^{2\pi} d\phi \frac{\sin\eta}{(\cosh\mu_0 - \cos\eta)} \left[\frac{\partial V_o}{\partial \mu} \right]_{\mu=-\mu_0} \qquad (19)$$

Q_1 is given by the corresponding integral of charge density σ_1 on surface S_1 of sphere O_1. Since μ is decreasing outwards at S_1 (where $\mu = \mu_0$), the charge density σ_1 is

$$\sigma_1 = \frac{-\epsilon}{4\pi} \left(\frac{\partial V_o}{\partial n} \right)_1 = \left[\frac{1}{h_\mu} \cdot \frac{\partial V_o}{\partial \mu} \right]_{\mu=\mu_0} \qquad (20)$$

$(\partial V_o/\partial\mu)_{\mu=-\mu_0}$ is given by equation (A18) in which the three functions $J(\eta,-\mu_0)$, $K(\eta,-\mu_0)$, and $L(\eta,-\mu_0)$ are given by equations (A29) - (A31). Clearly the middle term in equation (A18) (proportional to $K(\eta,-\mu_0)$) makes no contribution to Q_2. To obtain $(\partial V_o/\partial\mu)_{\mu=\mu_0}$, we replace $-\mu_0$ by $+\mu_0$ on the right-hand side of equation (A18) and use the relations

$$J(\eta,\mu_0) = J(\eta,-\mu_0)\ ,\ K(\eta,\mu_0) = -K(\eta,-\mu_0)\ ,\ L(\eta,\mu_0) = -L(\eta,-\mu_0) \qquad (21)$$

These equations follow on noting that $(\partial V_o/\partial\mu)_{\mu=\mu_0}$ is obtained by replacing B_n by A_n in equation (A28). These results will be proven in Chapter III.

To calculate the right-hand member of equation (6), we use equation (A5) and the identity

$$\int_0^\pi \frac{\sin\eta P_n(\cos\eta)}{(\cosh\mu_0-\cos\eta)^{1/2}}\,d\eta = \sqrt{2}\sum_{m=0}^{\infty} e^{-(m+\frac{1}{2})\mu_0}\int_0^\pi \sin\eta P_n(\cos\eta)P_m(\cos\eta)d\eta$$

$$= \frac{2\sqrt{2}}{2n+1}e^{-(n+\frac{1}{2})\mu_0} \qquad (22)$$

On substituting equation (A18) for $(\partial V_o/\partial\mu)_{\mu=-\mu_0}$, and making use of the expressions for $J(\eta,-\mu_0)$ and $L(\eta,-\mu_0)$ in equations (A29) and (A31), and also of equations (20) and (21), we obtain expressions for both Q_2 and Q_1. It is convenient to introduce the notation used in the following chapter,

$$\alpha_0 = \frac{1}{2}\sum_{n=0}^{\infty}(2n+1)\frac{e^{-(n+\frac{1}{2})\mu_0}}{\sinh(n+\frac{1}{2})\mu_0}\ ,\quad \beta_2-\beta_1 = \frac{1}{2}\sum_{n=0}^{\infty}\frac{e^{-(n+\frac{1}{2})\mu_0}}{\cosh(n+\frac{1}{2})\mu_0} \qquad (23)$$

Then

$$Q_2 = -2\epsilon Ec^2\alpha_0\cos\psi + 2\epsilon\Phi_0 c(\beta_2-\beta_1) \tag{24}$$

$$Q_1 = 2\epsilon Ec^2\alpha_0\cos\psi + 2\epsilon\Phi_0 c(\beta_2-\beta_1) \tag{25}$$

These expressions are identical with those obtained in the following chapter when the spheres have the same radius and the same potential. We note that

$$Q_2 = 0 \quad \text{when} \quad \Phi_0 = \frac{Ec\alpha_0\cos\psi}{\beta_2-\beta_1} = \hat{\Phi}_0 \text{ , say,} \tag{26}$$

$$Q_1 = 0 \quad \text{when} \quad \Phi_0 = \frac{-Ec\alpha_0\cos\psi}{\beta_2-\beta_1} = -\hat{\Phi}_0 \tag{27}$$

Q_2 and Q_1 are expressed in terms of one independent variable Φ_0, which can assume negative and positive values. We assign to the two droplets a common potential Φ which can vary over the range $-\hat{\Phi}_0$ to Φ_0. Q_2 and Q_1 become linear functions of Φ, as given by equations (24) and (25), on putting $\Phi_0 = \Phi$. The energy of charging the droplets to a common potential Φ_0 is

$$W(Q_1,Q_2) = \int_0^{Q_1} \Phi dQ_1 + \int_0^{Q_2} \Phi dQ_2$$

$$= Q_1\Phi_0 - \int_0^{Q_1} Q_1 d\Phi + Q_2\Phi_0 - \int_0^{Q_2} Q_2 d\Phi \tag{28}$$

on integrating by parts. We wish to express this energy in terms of Q_1 and Q_2. It will be explained in the following chapter that it is not possible to

have the two droplets at the same potential Φ_0 at large separations without assuming very large values for Q_1 and Q_2. We must assume that the charges Q_1 and Q_2 have meaningful values. For such fixed charges, independent of separation, the interaction energy is the change in $W(Q_1,Q_2)$ as the two droplets are brought together to a given separation from infinity. In terms of Q_1 and Q_2,

$$(Q_1+Q_2)\Phi_0 - \frac{(Q_2+2\epsilon Ec^2\alpha_0\cos\psi)^2}{\epsilon c(\beta_2-\beta_1)} - \frac{(Q_1-2\epsilon Ec^2\alpha_0\cos\psi)^2}{\epsilon c(\beta_2-\beta_1)} \tag{29}$$

Also, by substituting equation (25) (with Φ_0 replaced by Φ) for Q_1 and using equation (26) for $\hat{\Phi}_0$, we readily obtain

$$-\int_{Q_1=0}^{Q_1} Q_1 d\Phi - -\int_{-\hat{\Phi}_0}^{\Phi_0} Q_1 d\Phi - \frac{-Q_1^2}{4\epsilon c(\beta_2-\beta_1)} \tag{30}$$

Similarly

$$-\int_{Q_2=0}^{Q_2} Q_2 d\Phi - -\int_{\hat{\Phi}_0}^{\Phi_0} Q_2 d\Phi - \frac{-Q_2^2}{4\epsilon c(\beta_2-\beta_1)} \tag{31}$$

On adding equations (29) - (31), after some rearrangement,

$$W(Q_1,Q_2) - \frac{1}{4\epsilon c(\beta_2-\beta_1)}\left[Q_1^2 + Q_2^2 - 16\epsilon^2E^2c^4\alpha_0^2\cos^2\psi\right] \tag{32}$$

It is readily verified that equation (32) is obtained more directly from the first relation in equation (28) by using the relations

$$\Phi = \frac{Q_2}{2\epsilon c(\beta_2-\beta_1)} + \hat{\Phi}_0 = \frac{Q_1}{2\epsilon c(\beta_2-\beta_1)} - \hat{\Phi}_0 \tag{33}$$

In the following chapter, $W(Q_1,Q_2)$ is given by equation (III.45) which is derived differently for the general case of unequal spheres by treating Q_1 and Q_2 as independent variables. On substituting equations (24) and (25) into equation (III.45) and eliminating Φ_0, we obtain for two spheres of equal radius ($\mu_1 = \mu_2 = \mu_0$), after rearrangement,

$$W(Q_1,Q_2) = \frac{1}{4\epsilon c(\beta_2-\beta_1)}\left[Q_1^2 + Q_2^2 - \frac{\beta_2}{\beta_2+\beta_1}16\epsilon^2E^2c^4\alpha_0^2\cos^2\psi\right] \tag{34}$$

where, by equation (III.47), $\beta_2 > \beta_1 > 0$. The formulae in equations (32) and (34) are not identical. In equation (34), Q_1 and Q_2 are regarded as independent variables, whereas in equation (32) they are related through the parameter Φ_0, so that

$$Q_2 = Q_1 - 4\epsilon Ec^2\alpha_0\cos\psi \tag{35}$$

We demonstrate as follows that the charging processes are different. The charging process which yields equation (III.45), as described by equations (III.43) and (III.44), is illustrated in the (Q_1,Q_2) plane in Figure 2a. The process starts at the origin, $Q_1 = Q_2 = 0$, proceeds first along the Q_1-axis, and then parallel to the Q_2-axis. It is only at the very end of this path that the potentials of the two spheres, Φ_1 and Φ_2, as defined in equations (III.39) and (III.40), become identical and equal to Φ_0. The charging process on which equation (32) is based proceeds along the common line in the (Q_1,Q_2)-plane illustrated in Figure 3b, but the starting

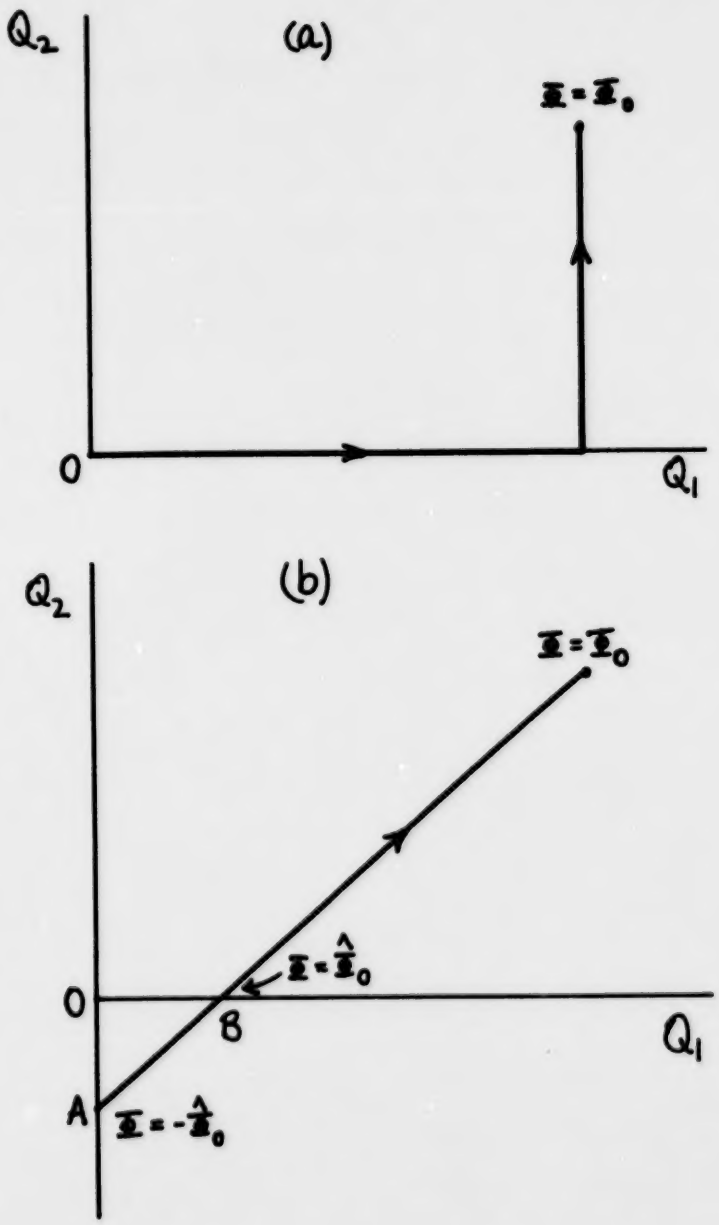

Figure 2. Graphical illustrations of the two methods of charging which lead to (a) equation (III.45), and (b) equation (32). In (b), $OA = OB = 4\epsilon Ec^2 a_0 \cos\psi$.

points (A and B) are different for the two spheres. By equation (34), Q_1 and Q_2 cannot be zero simultaneously, since it is impossible for the two spheres to be uncharged and to be at the same potential in the presence of an applied electric field. Thus, equation (32) is not the conventional energy of charging. It is quite possible to replace the portion AB of the common line in Figure 3b by OB, thereby achieving the same starting point of the charging process, but the potentials of the two sphres will be different along OB. Both equations (32) and (34) are unsuitable at large separations R, since in their last terms $c^4\alpha_0^2 \sim R^2$ at large R. In equations (24) and (25), $c^2\alpha_0 \sim R$ and in equations (26) and (27), $\hat{\Phi}_0 \sim R$ at large R. The whole approach described above for determining the energy of charging breaks down at large separations R. These results are consistent with our contention in Chapter III that one should avoid the model of equal potentials for the two spheres when calculating the interaction energy.

APPENDIX A

Derivation of Equations (12) and (13)

Similarly to our treatment of the two spheres in contact in terms of tangent-sphere coordinates, the constants in equation (11) are determined by superimposing the parts independent of ϕ and proportional to $\cos\phi$. We assume the two spheres to have equal potentials Φ_0, so that $V_o(\mu_0,\eta,\phi) = V_o(-\mu_0,\eta,\phi) = \Phi_0$. Hence, considering the terms independent of ϕ

$$\Phi_0 = -\frac{E c \sinh\mu_0 \cos\psi}{(\cosh\mu_0-\cos\eta)}$$

$$+ (\cosh\mu_0-\cos\eta)^{1/2} \sum_{n=0}^{\infty} {}_6A_n e^{(n+\frac{1}{2})\mu_0} + B_n e^{-(n+\frac{1}{2})\mu_0} {}_7P_n(\cos\eta) \qquad (A1)$$

$$\Phi_0 = \frac{E c \sinh\mu_0 \cos\psi}{(\cosh\mu_0-\cos\eta)}$$

$$+ (\cosh\mu_0-\cos\eta)^{1/2} \sum_{n=0}^{\infty} {}_6A_n e^{-(n+\frac{1}{2})\mu_0} + B_n e^{(n+\frac{1}{2})\mu_0} {}_7P_n(\cos\eta) \qquad (A2)$$

The sum and difference of equations (A1) and (A2) yield, respectively

$$\frac{\Phi_0}{(\cosh\mu_0-\cos\eta)^{1/2}} = \sum_{n=0}^{\infty} (A_n+B_n)\cosh(n+\tfrac{1}{2})\mu_0 P_n(\cos\eta) \qquad (A3)$$

$$\frac{E c \sinh\mu_0 \cos\psi}{(\cosh\mu_0-\cos\eta)^{1/2}} = \sum_{n=0}^{\infty} (A_n-B_n)\sinh(n+\tfrac{1}{2})\mu_0 P_n(\cos\eta) \qquad (A4)$$

We now use the expansion

$$(\cosh\mu_0-\cos\eta)^{-1/2} = \sqrt{2}\sum_{n=0}^{\infty} e^{-(n+\frac{1}{2})\mu_0}P_n(\cos\eta) \tag{A5}$$

and its derivative with respect to μ_0

$$\frac{\sinh\mu_0}{2(\cosh\mu_0-\cos\eta)^{3/2}} = \sqrt{2}\sum_{n=0}^{\infty}(n+\tfrac{1}{2})e^{-(n+\frac{1}{2})\mu_0}P_n(\cos\eta) \tag{A6}$$

Hence, equating coefficients of $P_n(\cos\eta)$

$$(A_n+B_n)\cosh(n+\tfrac{1}{2})\mu_0 = \sqrt{2}\Phi_0 e^{-(n+\frac{1}{2})\mu_0} \tag{A7}$$

$$(A_n-B_n)\sinh(n+\tfrac{1}{2})\mu_0 = 2\sqrt{2}Ec(n+\tfrac{1}{2})\cos\psi e^{-(n+\frac{1}{2})\mu_0} \tag{A8}$$

from which we derive

$$\begin{matrix}A_n\\B_n\end{matrix} = \frac{1}{\sqrt{2}}e^{-(n+\frac{1}{2})\mu_0}\left[\frac{\Phi_0}{\cosh(n+\frac{1}{2})\mu_0} \pm \frac{2Ec(n+\frac{1}{2})\cos\psi}{\sinh(n+\frac{1}{2})\mu_0}\right] \tag{A9}$$

We now consider the terms in equation (11) which are proportional to $\cos\phi$. Putting $\mu = \mu_0$ and $\mu = -\mu_0$, and noting that Φ_0 is independent of ϕ, we obtain the boundary conditions

$$0 = \frac{-E c \sin\eta \sin\psi}{(\cosh\mu_0 - \cos\eta)}$$

$$+ (\cosh\mu_0 - \cos\eta)^{1/2} \sum_{n=0}^{\infty} \left[C_n e^{(n+\frac{1}{2})\mu_0} + D_n e^{-(n+\frac{1}{2})\mu_0} \right] P_n^1(\cos\eta) \qquad \text{(A10)}$$

$$0 = \frac{-E c \sin\eta \sin\psi}{(\cosh\mu_0 - \cos\eta)}$$

$$+ (\cosh\mu_0 - \cos\eta)^{1/2} \sum_{n=0}^{\infty} \left[C_n e^{-(n+\frac{1}{2})\mu_0} + D_n e^{(n+\frac{1}{2})\mu_0} \right] P_n^1(\cos\eta) \qquad \text{(A11)}$$

Adding and subtracting,

$$0 = \frac{-E c \sin\eta \sin\psi}{(\cosh\mu_0 - \cos\eta)} + (\cosh\mu_0 - \cos\eta)^{1/2} \sum_{n=0}^{\infty} (C_n + D_n) \cosh(n+\tfrac{1}{2})\mu_0 P_n^1(\cos\eta) \qquad \text{(A12)}$$

$$0 = + (\cosh\mu_0 - \cos\eta)^{1/2} \sum_{n=0}^{\infty} (C_n - D_n) \sinh(n+\tfrac{1}{2})\mu_0 P_n^1(\cos\eta) \qquad \text{(A13)}$$

so that $C_n = D_n$ and

$$\frac{E c \sin\eta \sin\psi}{(\cosh\mu_0 - \cos\eta)^{3/2}} = 2 \sum_{n=0}^{\infty} C_n \cosh(n+\tfrac{1}{2})\mu_0 P_n^1(\cos\eta) \qquad \text{(A14)}$$

On differentiating equation (A5) with respect to η

$$\tfrac{1}{2}(\cosh\mu_0 - \cos\eta)^{-3/2} \sin\eta = \sqrt{2} \sum_{n=0}^{\infty} e^{-(n+\frac{1}{2})\mu_0} P_n^1(\cos\eta) \qquad \text{(A15)}$$

Equating coefficients of $P_n^1(\cos\eta)$ in equations (A14) and (A15),

$$C_n = D_n = \sqrt{2}Ec\sin\psi \frac{e^{-(n+\frac{1}{2})\mu_0}}{\cosh(n+\frac{1}{2})\mu_0} \tag{A16}$$

Derivation of Equation (18)

In equation (17) (with $\mu_0 > 0$)

$$\left[\frac{\partial V_Q}{\partial \mu}\right]_{\mu=-\mu_0} = \frac{Ec}{(\cosh\mu_0-\cos\eta)^2}\left[\cos\psi(\cosh\mu_0\cos\eta-1) - \sin\psi\sin\eta\sinh\mu_0\cos\phi\right]$$

$$- \frac{\sinh\mu_0}{2(\cosh\mu_0-\cos\eta)^{1/2}} \sum_{n=0}^{\infty}\left[A_n e^{-(n+\frac{1}{2})\mu_0} + B_n e^{-(n+\frac{1}{2})\mu_0}\right]P_n(\cos\eta)$$

$$+ (\cosh\mu_0-\cos\eta)^{1/2} \sum_{n=0}^{\infty}(n+\tfrac{1}{2})\left[A_n e^{-(n+\frac{1}{2})\mu_0} - B_n e^{-(n+\frac{1}{2})\mu_0}\right]P_n(\cos\eta)$$

$$- 2\cos\phi\left[\frac{\sinh\mu_0}{2(\cosh\mu_0-\cos\eta)^{1/2}} \sum_{n=1}^{\infty} C_n\cosh(n+\tfrac{1}{2})\mu_0 P_n^1(\cos\eta)\right.$$

$$\left. + (\cosh\mu_0-\cos\eta)^{1/2} \sum_{n=1}^{\infty} C_n(n+\tfrac{1}{2})\sinh(n+\tfrac{1}{2})\mu_0 P_n^1(\cos\eta)\right] \tag{A17}$$

On substituting for the three constants A_n, B_n, and C_n from equations (A9) and (A16) into equation (A17), we obtain

$$\left[\frac{\partial V_Q}{\partial \mu}\right]_{\mu=-\mu_0} = EcJ(\eta,-\mu_0)\cos\psi + EcK(\eta,-\mu_0)\cos\phi\sin\psi + \Phi_0 L(\eta,-\mu_0) \tag{A18}$$

where

$$J(\eta,-\mu_0) = \frac{\cosh\mu_0\cos\eta-1}{(\cosh\mu_0-\cos\eta)^2} + \sqrt{2}\sum_{n=0}^{\infty}(n+\tfrac{1}{2})e^{-(n+\frac{1}{2})\mu_0}P_n(\cos\eta)(\cosh\mu_0-\cos\eta)^{1/2}$$

$$\times\left[\frac{\sinh\mu_0}{(\cosh\mu_0-\cos\eta)} + (2n+1)\coth(n+\tfrac{1}{2})\mu_0\right] \qquad \text{(A19)}$$

$$K(\eta,-\mu_0) = \frac{-\sinh\mu_0\sin\eta}{(\cosh\mu_0-\cos\eta)^2} - \sqrt{2}\sum_{n-1}^{\infty}e^{-(n+\frac{1}{2})\mu_0}P_n^1(\cos\eta)(\cosh\mu_0-\cos\eta)^{1/2}$$

$$\times\left[\frac{\sinh\mu_0}{(\cosh\mu_0-\cos\eta)} + (2n+1)\tanh(n+\tfrac{1}{2})\mu_0\right] \qquad \text{(A20)}$$

$$L(\eta,-\mu_0) = -\frac{1}{\sqrt{2}}\sum_{n=0}^{\infty}e^{-(n+\frac{1}{2})\mu_0}P_n(\cos\eta)(\cosh\mu_0-\cos\eta)^{1/2}$$

$$\times\left[\frac{\sinh\mu_0}{(\cosh\mu_0-\cos\eta)} + (2n+1)\tanh(n+\tfrac{1}{2})\mu_0\right] \qquad \text{(A21)}$$

On substituting equation (A18) into equation (17), we get

$$F_z = \tfrac{\varepsilon}{4}(Ec)^2\cos^2\psi\int_0^{\pi}J^2(\eta,-\mu_0)\,\frac{\sin\eta(\cosh\mu_0\sin\eta-1)}{(\cosh\mu_0-\cos\eta)}\,d\eta$$

$$+\tfrac{\varepsilon}{8}(Ec)^2\sin^2\psi\int_0^{\pi}K^2(\eta,-\mu_0)\,\frac{\sin\eta(\cosh\mu_0\sin\eta-1)}{(\cosh\mu_0-\cos\eta)}\,d\eta$$

$$+\tfrac{\varepsilon}{4}\Phi_0^2\int_0^{\pi}L^2(\eta,-\mu_0)\,\frac{\sin\eta(\cosh\mu_0\sin\eta-1)}{(\cosh\mu_0-\cos\eta)}\,d\eta$$

$$+ 2Ec\Phi_0\cos\psi \int_0^{\pi} J(\eta,-\mu_0)L(\eta,-\mu_0) \frac{\sin\eta(\cosh\mu_0\sin\eta-1)}{(\cosh\mu_0-\cos\eta)} d\eta \qquad (A22)$$

By making use of equation (A6), we can express the coefficients $J(\eta,-\mu_0)$ and $K(\eta,-\mu_0)$ in equations (A19) and (A20) as

$$J(\eta,-\mu_0) = \frac{2\cosh\mu_0\cos\eta-\cosh^2\mu_0-1}{2(\cosh\mu_0-\cos\eta)^2}$$

$$+ 2\sqrt{2}(\cosh\mu_0-\cos\eta)^{1/2} \sum_{n=0}^{\infty} (n+\tfrac{1}{2})e^{-(n+\frac{1}{2})\mu_0}\coth(n+\tfrac{1}{2})\mu_0 P_n(\cos\eta)$$

$$\times \left[2(n+\tfrac{1}{2})\coth(n+\tfrac{1}{2})\mu_0 - 2\coth\mu_0 - \frac{\sinh\mu_0}{(\cosh\mu_0-\cos\eta)}\right] \qquad (A23)$$

or, alternatively, as

$$J(\eta,-\mu_0) = \sqrt{2}(\cosh\mu_0-\cos\eta)^{1/2} \sum_{n=0}^{\infty} (n+\tfrac{1}{2})e^{-(n+\frac{1}{2})\mu_0} P_n(\cos\eta)$$

$$\times \left[2(n+\tfrac{1}{2})\coth(n+\tfrac{1}{2})\mu_0 - 2\coth\mu_0 - \frac{\sinh\mu_0}{(\cosh\mu_0-\cos\eta)}\right] \qquad (A24)$$

On differentiating equation (A5) with respect to η, we get

$$\frac{\sin\eta}{2(\cosh\mu_0-\cos\eta)^{3/2}} = \sqrt{2} \sum_{n=1}^{\infty} e^{-(n+\frac{1}{2})\mu_0} P_n^1(\cos\eta) \qquad (A25)$$

which allows us to express equation (A20) as

$$K(\eta,-\mu_0) = \frac{3}{2}\frac{\sin\eta}{(\cosh\mu_0-\cos\eta)^2}$$

$$- 2\sqrt{2}\sum_{n=1}^{\infty}(n+\tfrac{1}{2})e^{-(n+\frac{1}{2})\mu_0}\tanh(n+\tfrac{1}{2})\mu_0 P_n^1(\cos\eta)(\cosh\mu_0-\cos\eta)^{1/2} \qquad \text{(A26)}$$

or, alternatively, as

$$K(\eta,-\mu_0) = -\sqrt{2}(\cosh\mu_0-\cos\eta)^{1/2}\sum_{n=0}^{\infty}e^{-(n+\frac{1}{2})\mu_0}P_n^1(\cos\eta)$$

$$\times\left[2(n+\tfrac{1}{2})\tanh(n+\tfrac{1}{2})\mu_0 + \frac{3\sinh\mu_0}{(\cosh\mu_0-\cos\eta)}\right] \qquad \text{(A27)}$$

Our formula for $V_o(\mu,\eta,\phi)$ in equation (11), with the constants A_n, B_n, and C_n given in equations (A9) and (A16), is identical with the corresponding one of Davis [13] when his two spheres are assumed to be identical (same radius a and same potential Φ_0). However, our result for $(\partial V_o/\partial\mu)_{\mu=-\mu_0}$ appears to differ from his, which is

$$\left[\frac{\partial V_o}{\partial\mu}\right]_{\mu=-\mu_0} = (\cosh\mu_0-\cos\eta)^{1/2}\left[\sum_{n=0}^{\infty}(2n+1)e^{(n+\frac{1}{2})\mu_0}B_nP_n(\cos\eta)\right.$$

$$\left. + \cos\phi\sum_{n=1}^{\infty}(2n+1)e^{(n+\frac{1}{2})\mu_0}C_nP_n^1(\cos\eta)\right] \qquad \text{(A28)}$$

so that, on substituting for A_n, B_n, and C_n,

$$J(\eta,-\mu_0) = (\cosh\mu_0-\cos\eta)^{1/2}2\sqrt{2}\sum_{n=0}^{\infty}\frac{(n+\frac{1}{2})^2}{\sinh(n+\frac{1}{2})\mu_0}P_n(\cos\eta) \tag{A29}$$

$$K(\eta,-\mu_0) = (\cosh\mu_0-\cos\eta)^{1/2}2\sqrt{2}\sum_{n=0}^{\infty}\frac{n+\frac{1}{2}}{\cosh(n+\frac{1}{2})\mu_0}P_n(\cos\eta) \tag{A30}$$

$$L(\eta,-\mu_0) = -(\cosh\mu_0-\cos\eta)^{1/2}\sqrt{2}\sum_{n=0}^{\infty}\frac{n+\frac{1}{2}}{\cosh(n+\frac{1}{2})\mu_0}P_n(\cos\eta) \tag{A31}$$

It is possible to prove the equivalence of our expressions for $J(\eta,-\mu_0)$, $K(\eta,-\mu_0)$, and $L(\eta,-\mu_0)$ with those of Davis [13].

III. INTERACTION OF TWO CONDUCTING SPHERICAL DROPLETS IN A UNIFORM ELECTRIC FIELD: UNEQUAL SPHERES

We consider here the general case of two unequal, perfectly conducting, spherical droplets of radii R_1 and R_2, and potentials Φ_1 and Φ_2. The surfaces of the droplets O_1 and O_2 are defined by bipolar coordinates $\mu = \mu_1$ and $\mu = -\mu_2$, respectively, where $\mu_1 > 0$ and $\mu_2 > 0$ (see Figure 3). Consider the sphere represented by $\mu = \mu_1$. It has a radius $R_1 = c/\sinh\mu_1$ and since $x = y = 0$ at its centre, it follows from equation (III.1) that $\eta = 0$ and thus $z = D_1 = c\cdot\coth\mu_1$ at its centre. Eliminating c, we get the quadratic

$$R_1\cosh\mu_1 = \tfrac{1}{2}R_1(e^{\mu_1} + e^{-\mu_1}) = D_1 \qquad (1)$$

which has a solution

$$e^{\mu_1} = \frac{1}{R_1}\left[D_1+(D_1^{\,2}-R_1^{\,2})^{1/2}\right] \quad \text{or} \quad \mu_1 = \ln\left[\frac{D_1+(D_1^{\,2}-R_1^{\,2})^{1/2}}{R_1}\right] \qquad (2)$$

We need to take the positive sign in the solution of the quadratic because $R_1 = 0$ and $z = c = D_1$ when $\mu_1 = \infty$. Similarly for the sphere $\mu = -\mu_2$, $R_2 = c/\sinh\mu_2$, $z = -D_2 = -c\coth\mu_2$, and so $\mu_2 = \ln\left[[D_2+(D_2^{\,2}-R_2^{\,2})^{1/2}]/R_2\right]$. (Also $R = O_1O_2 = D_1 + D_2$.) We note that the two droplets cannot touch for unequal potentials Φ_1 and Φ_2.

The solution of Laplace's equation for the potential V_o in the oil medium is given by equation (II.11) where now the constants are given by (see Appendix B for a derivation)

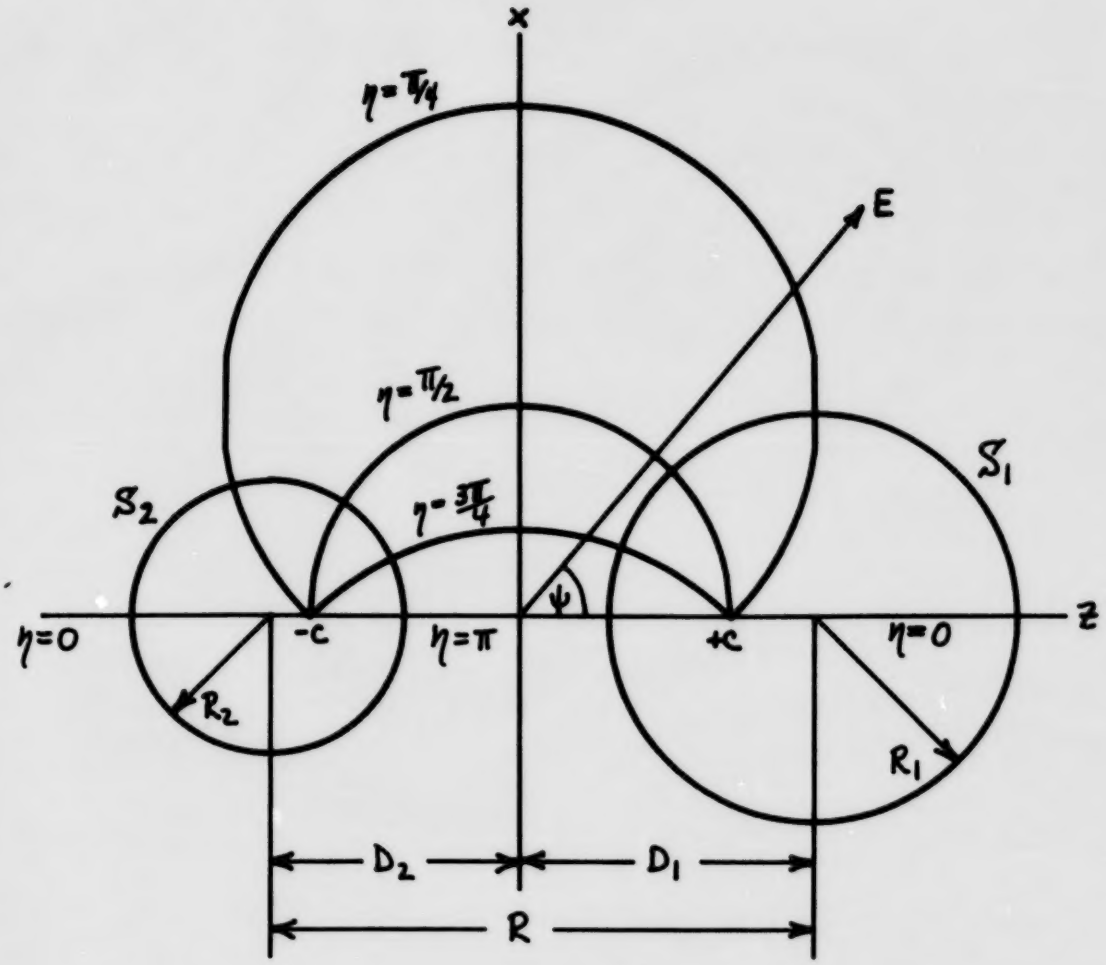

Figure 3. Bipolar coordinates for two unequal spheres at separation R. Spheres S_1 and S_2 have coordinates $\mu = \mu_1$ and $\mu = -\mu_2$, respectively. The values of the parameters used in this Figure are $R_1 = 1.5$, $R_2 = 1$, $R = 4$, so that $D_1 = 2.156$, $D_2 = 1.844$, $c = 1.549$, $\mu_1 = 0.904$, $\mu_2 = 1.222$.

$$A_n = \frac{\sqrt{2}Ec\cos\psi(2n+1)\left[e^{(2n+1)\mu_2}+1\right] + \sqrt{2}\left[\Phi_1 e^{(2n+1)\mu_2}-\Phi_2\right]}{e^{(2n+1)(\mu_1+\mu_2)} - 1} \tag{3}$$

$$B_n = \frac{-\sqrt{2}Ec\cos\psi(2n+1)\left[e^{(2n+1)\mu_1}+1\right] + \sqrt{2}\left[\Phi_2 e^{(2n+1)\mu_1}-\Phi_1\right]}{e^{(2n+1)(\mu_1+\mu_2)} - 1} \tag{4}$$

$$C_n = \frac{2\sqrt{2}Ec\sin\psi\left[e^{(2n+1)\mu_2}-1\right]}{e^{(2n+1)(\mu_1+\mu_2)} - 1} \tag{5}$$

$$D_n = \frac{2\sqrt{2}Ec\sin\psi\left[e^{(2n+1)\mu_1}-1\right]}{e^{(2n+1)(\mu_1+\mu_2)} - 1} \tag{6}$$

These expressions for A_n, B_n, C_n, and D_n are identical with those quoted by Davis [13].

Charge Densities

We wish to determine the charge densities on the two perfectly conducting spheres whose surfaces we denote by S_1 and S_2. The charge density on the surface S_2 of sphere O_2 ($\mu = -\mu_2$) is

$$\sigma_2 = -\frac{\epsilon}{4\pi}\left[\frac{\partial V_o}{\partial n}\right]_{\mu=-\mu_2} = -\frac{\epsilon}{4\pi}\left[\frac{1}{h_\mu}\frac{\partial V_o}{\partial \mu}\right]_{\mu=-\mu_2} = -\frac{\epsilon}{4\pi}\frac{(\cosh\mu_2-\cos\eta)}{c}\left[\frac{\partial V_o}{\partial \mu}\right]_{\mu=-\mu_2} \tag{7}$$

noting that μ is increasing outwards from S_2. Evaluating the derivative yields (see Appendix B),

$$\left[\frac{\partial V_o(\mu,\eta,\phi)}{\partial \mu}\right]_{\mu=-\mu_2} = -(\cosh\mu_2-\cos\eta)^{1/2}\left[\sum_{n=0}^{\infty}(2n+1)B_n e^{(n+\frac{1}{2})\mu_2}P_n(\cos\eta)\right.$$

$$\left. + \cos\phi \sum_{n=1}^{\infty}(2n+1)D_n e^{(n+\frac{1}{2})\mu_2}P_n^1(\cos\eta)\right] \qquad (8)$$

The charge density σ_1 on the surface S_1 of sphere O_1 is

$$\sigma_1 = \frac{\epsilon}{4\pi}\left[\frac{1}{h_\mu}\frac{\partial V_o}{\partial \mu}\right]_{\mu=\mu_1} = \frac{\epsilon}{4\pi}\frac{(\cosh\mu_1-\cos\eta)}{c}\left[\frac{\partial V_o}{\partial \mu}\right]_{\mu=\mu_1} \qquad (9)$$

noting that now μ is decreasing outwards at S_1, where $\mu = \mu_1$. The derivative is given by (see Appendix B)

$$\left[\frac{\partial V_o(\mu,\eta,\phi)}{\partial \mu}\right]_{\mu=\mu_1} = (\cosh\mu_1-\cos\eta)^{1/2}\left[\sum_{n=0}^{\infty}(2n+1)A_n e^{(n+\frac{1}{2})\mu_1}P_n(\cos\eta)\right.$$

$$\left. + \cos\phi \sum_{n=1}^{\infty}(2n+1)C_n e^{(n+\frac{1}{2})\mu_1}P_n^1(\cos\eta)\right] \qquad (10)$$

Equations (8) and (10) are the same as those quoted by Davis.

We now consider special cases of equations (8) and (10). Suppose that the radii R_1 and R_2 are equal, but the potentials Φ_1 and Φ_2 are different. We write $\mu_1 = \mu_2 = \mu_0$, say. Then equations (3) and (4) can be written as

$$A_n = A_n^{(1)} + A_n^{(2)}, \quad B_n = B_n^{(1)} + B_n^{(2)} \qquad (11)$$

where

$$A_n^{(1)} = \frac{1}{\sqrt{2}} \frac{\left[\Phi_1 - \Phi_2 e^{-(2n+1)\mu_0}\right]}{\sinh(2n+1)\mu_0}, \quad B_n^{(1)} = \frac{1}{\sqrt{2}} \frac{\left[\Phi_2 - \Phi_1 e^{-(2n+1)\mu_0}\right]}{\sinh(2n+1)\mu_0} \tag{12}$$

$$A_n^{(2)} = -B_n^{(2)} = \frac{\sqrt{2}Ec(n+\frac{1}{2})e^{-(n+\frac{1}{2})\mu_0}\cos\psi}{\sinh(n+\frac{1}{2})\mu_0} \tag{13}$$

Similarly, equations (5) and (6) simplify to

$$C_n = D_n = \frac{\sqrt{2}Ec\sin\psi e^{-(n+\frac{1}{2})\mu_0}}{\cosh(n+\frac{1}{2})\mu_0} \tag{14}$$

It follows that we can write equations (7) and (8) as

$$-\frac{4\pi}{\epsilon}\frac{c\sigma_2}{(\cosh\mu_0-\cos\eta)} = \left[\frac{\partial V_o(\mu,\eta,\phi)}{\partial\mu}\right]_{\mu=-\mu_0}$$

$$= EcJ(\eta,-\mu_0)\cos\psi + EcK(\eta,-\mu_0)\cos\phi\sin\psi + \Phi_1 L_1(\eta,-\mu_0) + \Phi_2 L_2(\eta,-\mu_0) \tag{15}$$

where

$$J(\eta,-\mu_0) = 2\sqrt{2}(\cosh\mu_0-\cos\eta)^{1/2} \sum_{n=0}^{\infty} \frac{(n+\frac{1}{2})^2 P_n(\cos\eta)}{\sinh(n+\frac{1}{2})\mu_0} \tag{16}$$

$$K(\eta,-\mu_0) = -2\sqrt{2}(\cosh\mu_0-\cos\eta)^{1/2} \sum_{n=1}^{\infty} \frac{(n+\frac{1}{2})P_n^1(\cos\eta)}{\cosh(n+\frac{1}{2})\mu_0} \tag{17}$$

$$L_1(\eta,-\mu_0) = \sqrt{2}(\cosh\mu_0-\cos\eta)^{1/2} \sum_{n=0}^{\infty} \frac{(n+\frac{1}{2})e^{-(n+\frac{1}{2})\mu_0}}{\sinh(2n+1)\mu_0} P_n(\cos\eta) \tag{18}$$

$$L_2(\eta,-\mu_0) = -\sqrt{2}(\cosh\mu_0-\cos\eta)^{1/2} \sum_{n=0}^{\infty} \frac{(n+\frac{1}{2})e^{+(n+\frac{1}{2})\mu_0}}{\sinh(2n+1)\mu_0} P_n(\cos\eta) \tag{19}$$

If we assume equal potentials for the two spheres and write $\Phi_1 = \Phi_2 = \Phi_0$, then the last two terms in equation (15) are replaced by

$$\Phi_0 L(\eta,-\mu_0) = -\sqrt{2}\Phi_0(\cosh\mu_0-\cos\eta)^{1/2} \sum_{n=0}^{\infty} \frac{(n+\frac{1}{2})P_n(\cos\eta)}{\cosh(n+\frac{1}{2})\mu_0} \tag{20}$$

where $L = L_1 + L_2$, whereas $J(\eta,-\mu_0)$ and $K(\eta,-\mu_0)$ remain unchanged.

Consider now the surface charge σ_1 on sphere S_1 where $\mu = \mu_0$. We can write equations (9) and (10) for equal radii R_1 and R_2 as

$$\frac{4\pi}{\epsilon} \frac{c\sigma_1}{(\cosh\mu_0-\cos\eta)} = \left[\frac{\partial V_o(\mu,\eta,\phi)}{\partial\mu}\right]_{\mu=\mu_0}$$

$$= EcJ(\eta,\mu_0)\cos\psi + EcK(\eta,\mu_0)\cos\phi\sin\psi + \Phi_1 L_1(\eta,\mu_0) + \Phi_2 L_2(\eta,\mu_0) \tag{21}$$

On comparing equation (8) with (10) and making use of the relations in equations (11) - (14),

$$J(\eta,\mu_0) = J(\eta,-\mu_0) \ , \quad K(\eta,\mu_0) = -K(\eta,-\mu_0) \tag{22}$$

$$L_1(\eta,\mu_0) = -L_2(\eta,-\mu_0) \ , \quad L_2(\eta,\mu_0) = -L_1(\eta,-\mu_0) \tag{23}$$

and when $\Phi_1 = \Phi_2 = \Phi_0$

$$L(\eta,\mu_0) = L_1(\eta,\mu_0) + L_2(\eta,\mu_0) = -L(\eta,-\mu_0) \tag{24}$$

It might appear from the expressions for J, K, and L in equations (16) to (20) that we should have the opposite signs in the relations given in equations (22) to (24). A simple explanation of this apparent anomaly is obtained by considering the expression (II.11) for $V_o(\mu,\eta,\phi)$ in the particular case where $\mu_1 = \mu_2 = \mu_0$ and $\Phi_1 = \Phi_2 = \Phi_0$. Let us write now

$$V_o(\mu,\eta,\phi) = Ec\hat{J}(\eta,\mu)\cos\phi + Ec\hat{K}(\eta,\mu)\cos\phi\sin\phi + \Phi_0\hat{L}(\eta,\mu) \tag{25}$$

Bearing in mind that $A_n^{(2)} = -B_n^{(2)}$, $A_n^{(1)} = B_n^{(1)}$ and $C_n = D_n$, it is seen immediately that

$$\hat{J}(\eta,\mu) = -\hat{J}(\eta,-\mu) \ , \quad \hat{K}(\eta,\mu) = \hat{K}(\eta,-\mu) \ , \quad \hat{L}(\eta,\mu) = \hat{L}(\eta,-\mu) \tag{26}$$

Since $J(\eta,\mu)$, $K(\eta,\mu)$ and $L(\eta,\mu)$ are the the derivatives with respect to μ of $\hat{J}(\eta,\mu)$, $\hat{K}(\eta,\mu)$ and $\hat{L}(\eta,\mu)$, respectively, the relations in equations (22) and (24) follow. Indeed, the series in equations (16) to (20) diverge at $\mu_0 = 0$ as is illustrated schematically in Figure 4.

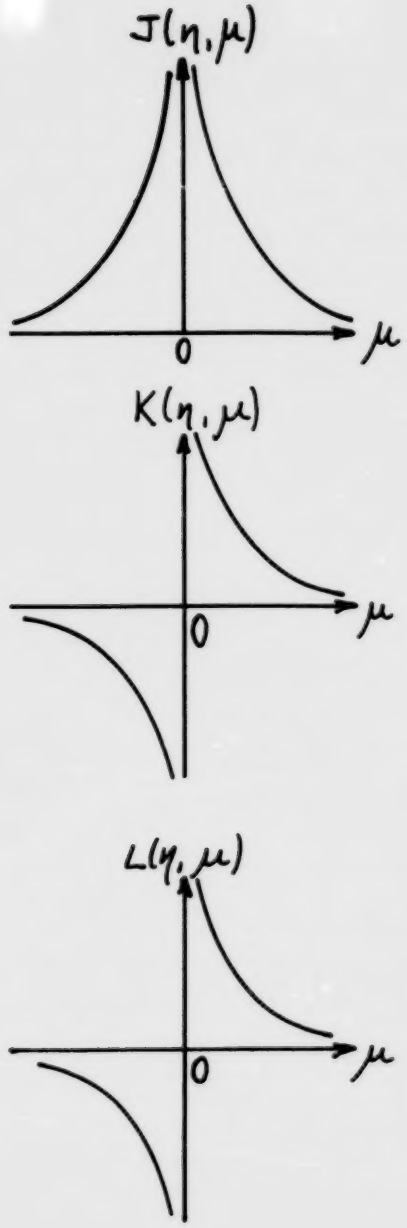

Figure 4. Schematic illustration of the functions $J(\eta,\mu)$, $K(\eta,\mu)$ and $L(\eta,\mu)$ defined by equations (16), (17), and (20).

Interaction Energy

Let us integrate the formula in equation (7) for σ_2, combined with equation (8), over the surface S_2 of sphere O_2. The total charge on the surface is

$$Q_2 = \int_{S_2} \sigma_2 dS_2 = \frac{-\epsilon}{4\pi} \int_0^{2\pi} d\phi \int_0^{\pi} d\eta \left[\frac{h_\eta h_\phi}{h_\mu} \frac{\partial V_o}{\partial \mu} \right]_{\mu=-\mu_2}$$

$$= -\frac{\epsilon c}{4\pi} \int_0^{2\pi} d\phi \int_0^{\pi} d\eta \frac{\sin\eta}{(\cosh\mu_2 - \cos\eta)} \left[\frac{\partial V_o}{\partial \mu} \right]_{\mu=-\mu_2} \tag{27}$$

since

$$h_\mu = h_\eta = \frac{c}{(\cosh\mu - \cos\eta)}, \quad h_\phi = \frac{c\sin\eta}{(\cosh\mu - \cos\eta)} \tag{28}$$

Clearly the term in $\cos\phi$ in equation (8) makes no contribution to Q_2, so that

$$Q_2 = \tfrac{1}{2}\epsilon c \sum_{n=0}^{\infty} (2n+1) B_n e^{(n+\frac{1}{2})\mu_2} \int_0^{\pi} d\eta \frac{\sin\eta \, P_n(\cos\eta)}{(\cosh\mu_2 - \cos\eta)^{1/2}} \tag{29}$$

With the use of equation (A5), the integral in equation (29) can be written as

$$\int_0^{\pi} d\eta \frac{\sin\eta \, P_n(\cos\eta)}{(\cosh\mu_2 - \cos\eta)^{1/2}} = \sqrt{2} \sum_{m=0}^{\infty} e^{-(m+\frac{1}{2})\mu_2} \int_0^{\pi} d\eta \, \sin\eta \, P_n(\cos\eta) P_m(\cos\eta)$$

$$- 2\sqrt{2}\,\frac{e^{-(n+\frac{1}{2})\mu_2}}{2n+1} \tag{30}$$

Hence, substituting equation (4) for B_n,

$$Q_2 = \sqrt{2}\epsilon c \sum_{n=0}^{\infty} B_n = -2\epsilon E c^2 \cos\psi\beta_0 - 2\epsilon c\Phi_1\beta_1 + 2\epsilon c\Phi_2\beta_2 \tag{31}$$

where

$$\beta_0 = \sum_{n=0}^{\infty} \frac{(2n+1)(e^{(2n+1)\mu_1} + 1)}{e^{(2n+1)(\mu_1+\mu_2)} - 1} \tag{32}$$

$$\beta_1 = \sum_{n=0}^{\infty} \frac{1}{e^{(2n+1)(\mu_1+\mu_2)} - 1} \tag{33}$$

$$\beta_2 = \sum_{n=0}^{\infty} \frac{e^{(2n+1)\mu_1}}{e^{(2n+1)(\mu_1+\mu_2)} - 1} \tag{34}$$

Integrating the formula in equation (9) for σ_1, combined with equation (10), over the surface S_1 of sphere O_1, the total charge on the surface is

$$Q_1 = \int_{S_2} \sigma_1 dS_1 = \frac{\epsilon c}{4\pi} \int_0^{2\pi} d\phi \int_0^{\pi} d\eta \, \frac{\sin\eta}{(\cosh\mu_1 - \cos\eta)} \left[\frac{\partial V_o}{\partial \mu}\right]_{\mu=\mu_1}$$

$$= \sqrt{2}\epsilon c \sum_{n=0}^{\infty} A_n \tag{35}$$

Hence, from equation (3) for A_n,

$$Q_1 = 2\epsilon Ec^2\cos\psi\alpha_0 + 2\epsilon c\Phi_1\alpha_1 - 2\epsilon c\Phi_2\beta_1 \tag{36}$$

where

$$\alpha_0 = \sum_{n=0}^{\infty} \frac{(2n+1)\left[e^{(2n+1)\mu_2} + 1\right]}{e^{(2n+1)(\mu_1+\mu_2)} - 1} \tag{37}$$

$$\alpha_1 = \sum_{n=0}^{\infty} \frac{e^{(2n+1)\mu_2}}{e^{(2n+1)(\mu_1+\mu_2)} - 1} \tag{38}$$

We can solve equations (31) and (36) to obtain

$$\Phi_1 = \lambda_1 Q_1 + \lambda_2 Q_2 + \lambda_3 Ec\cos\psi \tag{39}$$

$$\Phi_2 = \lambda_2 Q_1 + r_2 Q_2 + r_3 Ec\cos\psi \tag{40}$$

where

$$\lambda_1 = \frac{\beta_2}{2\epsilon c(\alpha_1\beta_2 - \beta_1^2)}, \quad \lambda_2 = \frac{\beta_1}{2\epsilon c(\alpha_1\beta_2 - \beta_1^2)}, \quad \lambda_3 = \frac{\beta_1\beta_0 - \alpha_0\beta_2}{\alpha_1\beta_2 - \beta_1^2} \tag{41}$$

$$r_2 = \frac{\alpha_1}{2\epsilon c(\alpha_1\beta_2 - \beta_1^2)}, \quad r_3 = \frac{\alpha_1\beta_0 - \alpha_0\beta_1}{\alpha_1\beta_2 - \beta_1^2} \tag{42}$$

If the two spheres are grounded, so that $\Phi_1 = \Phi_2 = 0$, then the first terms in equations (31) and (36), which are proportional to the applied field E, yield the charges on the droplets which are induced by E. It will be explained below that the spheres cannot both be at zero potential at large separations.

We proceed to determine the electrostatic energy of the two spheres due to their charges in the presence of the applied electric field. Imagine that the two droplets are completely discharged so that $Q_1 = Q_2 = 0$. If we now charge S_2 first, the energy of charging is

$$\int_0^{Q_2} \Phi_2(Q_1=0)dQ_2 = \tfrac{1}{2}\gamma_2 Q_2^2 + \gamma_3 E\cos\psi Q_2 \tag{43}$$

Keeping the charge on S_2 fixed at Q_2, we now charge S_1 to yield the energy

$$\int_0^{Q_1} \Phi_1(Q_2 \text{ fixed})dQ_1 = \tfrac{1}{2}\lambda_1 Q_1^2 + \lambda_2 Q_1 Q_2 + \lambda_3 E\cos\psi Q_1 \tag{44}$$

The total energy of charging is therefore

$$\tfrac{1}{2}\lambda_1 Q_1^2 + \tfrac{1}{2}\gamma_2 Q_2^2 + \lambda_2 Q_1 Q_2 + E\cos\psi(\gamma_3 Q_2 + \lambda_3 Q_1) \tag{45}$$

from which we shall have to subtract the corresponding energy when the two spheres are far apart to obtain the corresponding interaction energy. Of course, the same energy is obtained by charging S_1 first and then S_2. The energy of charging the two spheres at infinite separation is

$$\frac{1}{2}\epsilon\left[\frac{Q_1^2}{R_1^2}+\frac{Q_2^2}{R_2^2}\right]-\frac{1}{2\epsilon c^2}(Q_1^2\sinh^2\mu_1+Q_2^2\sinh^2\mu_2) \tag{46}$$

Davis [13] has terms in his force expression between the two droplets which correspond to equation (45), but he does not give explicit formulae because of lack of space. He also has a term which is missing in equation (45) which is proportional to E^2 and which is the force between two uncharged droplets under the influence of the applied electric field. The corresponding part of the interaction energy will be found in Chapter IV.

On putting $\mu_1 = \mu_2 = \mu_0$ for two spheres of equal radius,

$$\alpha_0-\beta_0=\sum_{n=0}^{\infty}\frac{(n+\frac{1}{2})e^{-(n+\frac{1}{2})\mu_0}}{\sinh(n+\frac{1}{2})\mu_0}\,,\quad \alpha_1-\beta_2=\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{\sinh(2n+1)\mu_0}\,,$$

$$\beta_1=\frac{1}{2}\sum_{n=0}^{\infty}\frac{e^{-(2n+1)\mu_0}}{\sinh(2n+1)\mu_0} \tag{47}$$

Hence $\lambda_1 = r_2$ in equations (39) and (40) and

$$r_3=-\lambda_3=\frac{\alpha_0}{\beta_1+\beta_2}>0 \tag{48}$$

where

$$\beta_1+\beta_2=\frac{1}{2}\sum_{n=0}^{\infty}\frac{e^{-(n+\frac{1}{2})\mu_0}}{\sinh(n+\frac{1}{2})\mu_0} \tag{49}$$

Suppose that the equal spheres have radii $R_1 = R_2 = a$ and carry equal charges $Q_1 = Q_2 = Q$. Then from equation (48), the term in $\cos\psi$ in equation (45) is zero and the energy of interaction is

$$(\lambda_1+\lambda_2)Q^2 - \frac{Q^2}{\epsilon a} = \frac{Q^2}{2\epsilon c(\alpha_1-\beta_1)} - \frac{Q^2}{\epsilon a} \tag{50}$$

From equation (47), we can expand

$$\alpha_1 - \beta_1 = e^{-\mu_0}\left[1 - e^{-\mu_0} + 2e^{-2\mu_0} - e^{-3\mu_0} + 2e^{-4\mu_0} + \ldots\right] \tag{51}$$

Also

$$c = a\sinh\mu_0 = \frac{a}{2}e^{\mu_0}(1 - e^{-2\mu_0}) \tag{52}$$

Hence

$$\lambda_1 + \lambda_2 = \frac{1}{\epsilon a}\left[1 + e^{-\mu_0} - e^{-3\mu_0} - e^{-4\mu_0} + \ldots\right] \tag{53}$$

From $\cosh\mu_0 = R/2a$, we readily obtain

$$e^{-\mu_0} = \frac{R}{2a}\left[1 - [1 - (2\frac{a}{R})^2]^{1/2}\right] = \frac{a}{R} + (\frac{a}{R})^3 + 2(\frac{a}{R})^5 + \ldots \tag{54}$$

and so

$$(\lambda_1+\lambda_2)Q^2 - \frac{Q^2}{\epsilon a} = \frac{Q^2}{\epsilon R}\left[1 - (\frac{a}{R})^3 + \ldots\right] \tag{55}$$

which is identical with the corresponding interaction obtained by the twin multipole expansion.

Two limiting cases are of particular interest. Suppose that the two spheres carry no net charges, so that $Q_1 = Q_2 = 0$. Then, from equations (39) and (40),

$$\Phi_1 = \lambda_3 E c \cos\psi \ , \quad \Phi_2 = \tau_3 E c \cos\psi \tag{56}$$

If, in addition, the spheres have equal radii,

$$\Phi_1 = -\Phi_2 = \frac{-\alpha_0}{\beta_1+\beta_2} E c \cos\psi \tag{57}$$

where α_0 and $\beta_1+\beta_2$ are given in equations (47) and (49) respectively. The potentials Φ_1 and Φ_2 are now equal in magnitude but opposite in sign, the two signs of the potentials being consistent with the direction of the applied electric field. The other limiting case is grounded spheres, when $\Phi_1 = \Phi_2 = 0$. Then from equations (6) and (11)

$$Q_1 = 2\epsilon E c^2 \alpha_0 \cos\psi \ , \quad Q_2 = -2\epsilon E c^2 \beta_0 \cos\psi \tag{58}$$

which are opposite in sign.

We now question whether it is possible to ground both spheres in an electric field at large separations R, because there should be a potential difference between the two spheres in the presence of an applied electric field. Suppose that the separation R between the two (unequal) spheres becomes large, but the radii R_1 and R_2 of the two spheres remain unchanged.

Then μ_1 and μ_2 become large. From $D_1 = R_1\cosh\mu_1$ and $D_2 = R_2\cosh\mu_2$, we have $D_1 \simeq \frac{1}{2}R_1\exp(\mu_1)$ and $D_2 \simeq \frac{1}{2}R_2\exp(\mu_2)$. Also $D_1 = c\coth\mu_1$ and $D_2 = c\coth\mu_2$, so that for large μ_1 and μ_2, $D_1 = c$, $D_2 = c$ and $D_1 + D_2 = R = 2c$. Since $c_1 = R_1\sinh\mu_1 = R_2\sinh\mu_2$, it follows that

$$c = \tfrac{1}{2}R_1 e^{\mu_1} = \tfrac{1}{2}R_2 e^{\mu_2} = \frac{R}{2} \tag{59}$$

Also from equations (32) and (37), for large μ_1 and μ_2

$$\beta_0 = e^{-\mu_2} = \frac{R_2}{R_1}\ , \quad \alpha_0 = e^{-\mu_1} = \frac{R_1}{R} \tag{60}$$

using equation (59). Equation (58) therefore becomes

$$Q_1 = \tfrac{1}{2}\epsilon E R R_1\cos\psi\ , \quad Q_2 = -\tfrac{1}{2}\epsilon E R R_2\cos\psi \tag{61}$$

so that $|Q_1|$ and $|Q_2| \to \infty$ as $R \to \infty$. However, one would expect that Q_1 and Q_2 would remain finite as R increases, since the charges should tend to the values corresponding to single (isolated) spheres. Consider a single conducting sphere of radius a, carrying a charge Q in a uniform electric field E. Introducing spherical coordinates r,θ, with origin at the centre of the sphere and θ the angle to the direction of the applied field, the potential has the form

$$\Psi(r,\theta) = \Psi_0 + \frac{Q}{\epsilon r} + \left(\frac{B}{r^2} - Er\right)\cos\theta \tag{62}$$

where Ψ_0 is an additive constant potential. If we now assume $\Psi(a,\theta) = 0$, then $B = Ea^3$ and $Q = -\epsilon a\Psi_0$, where we expect Q and therefore Ψ_0 to have physically meaningful values.

Further insight into the source of the large values that Q_1 and Q_2 appear to reach in equation (61) is obtained as follows. In principle, the potential can be chosen zero anywhere since, in a 'linear system', the potential is undetermined to an additive constant. Let us ground sphere 2, so that $\Phi_2 = 0$ and assume $\psi = 0$. Then equations (31) and (36) become

$$Q_1 = 2\epsilon Ec^2\alpha_0 + 2\epsilon c\Phi_1\alpha_1 \tag{63}$$

$$Q_2 = -2\epsilon Ec^2\beta_0 - 2\epsilon c\Phi_1\beta_1 \tag{64}$$

We now move sphere 1 away from sphere 2, so that $R \to \infty$. From equations (32), (33), (37), and (38), for large μ_1 and μ_2, $\alpha_1 \simeq \alpha_0$ and $\beta_1 \simeq \alpha_0\beta_0$. Hence using equation (59), equations (63) and (64) become, for large R,

$$Q_1 \simeq \tfrac{1}{2}\epsilon ERR_1 + \epsilon\Phi_1 R_1 \tag{65}$$

$$Q_2 \simeq -\tfrac{1}{2}\epsilon ERR_2 - \frac{R_2}{R}\epsilon\Phi_1 R_1 \tag{66}$$

Noting that we expect a potential drop of $\Phi_1 = -ER$ between the two spheres due to the electric field and that we can ignore the second term in Q_2, we obtain

$$Q_1 = -\tfrac{1}{2}\epsilon\Phi_1 R_1 \ , \quad Q_2 = \tfrac{1}{2}\epsilon\Phi_1 R_2 \tag{67}$$

But these are precisely the values of Q_1 and Q_2 in equation (61) on putting $\cos\psi = 1$ and $\Phi_1 = -ER$. The high values of Q_1 and Q_2 seem to be due to the high potential difference for large R. Of course Q_1 and Q_2 are not arbitrary and must have reasonable values, and Φ_1 and Φ_2 are fixed by equations (39) and (40). We give one example to show that by choosing an appropriate position for the zero potential, we can have Q_1 and Q_2 tending to zero with increasing R. Consider two spheres of equal radius a, so that $\mu_1 = \mu_2 = \mu_0$. Put $\psi = 0$, choose the zero potential on the line of centres midway between the spheres, and assume that $\Phi_1 = -ER/2$ and $\Phi_2 = ER/2$. Noting that from $\cosh\mu_0 = R/2a$,

$$c = a\sinh\mu_0 = a(\cosh^2\mu_0 - 1)^{1/2} = \frac{R}{2}[1 - 2(\frac{a}{R})^2 + \ldots] \tag{68}$$

and, using equation (54), equation (47) yields

$$\alpha_0 = \beta_0 = e^{-\mu_0}(1 + e^{-\mu_0} + \ldots) = \frac{a}{R}(1 + \frac{a}{R} + \ldots) \tag{69}$$

$$\alpha_1 = \beta_2 = e^{-\mu_0}(1 + 2e^{-2\mu_0} + \ldots) = \frac{a}{R} + \ldots \tag{70}$$

$$\beta_1 = e^{-2\mu_0}(1 + e^{-2\mu_0} + \ldots) = (\frac{a}{R})^2 + \ldots \tag{71}$$

The two leading terms in equations (31) and (36) are

$$Q_1 = \tfrac{1}{2}\epsilon ER^2 \cdot \frac{a}{R}(1 + \frac{1}{R}) - \tfrac{1}{2}\epsilon R \cdot \tfrac{1}{2}ER \cdot \frac{a}{R} - 2\epsilon \cdot \frac{R}{2} \cdot \tfrac{1}{2}ER(\frac{a}{R})^2 = 0 \tag{72}$$

$$Q_2 = -\tfrac{1}{2}\epsilon ER^2 \cdot \frac{a}{R}(1 + \frac{1}{R}) + \epsilon R \cdot \tfrac{1}{2}ER \cdot \frac{a}{R} + 2\epsilon \cdot \frac{R}{2} \cdot \tfrac{1}{2}ER(\frac{a}{R})^2 = 0 \tag{73}$$

The higher order terms in Q_1 and $Q_2 \to 0$ as $R \to \infty$.

The above examples illustrate our contention that there is a potential drop of ER (for $\cos\psi = 1$) over a large distance R. It is conceivable that at small separations Q_1 and Q_2 may be such as to offset the potential drop due to the electric field and so permit the two spheres to have equal potentials and, in particular, zero potentials. But this cannot be the case at large separations if Q_1 and Q_2 have reasonable values. Thus it is not meaningful to seek the energy change when two spheres at constant equal potentials are brought to separation R from infinite separation in the presence of an applied electric field.

APPENDIX B

Derivation of the Constants in Equation (II.11)

The constants in equation (II.11) are determined by superimposing the parts independent of ϕ and proportional to $\cos\phi$. Since the spheres $\mu = \mu_1$ and $\mu = -\mu_2$ are equipotentials, equal to Φ_1 and Φ_2 respectively, $V_o(\mu_1,\eta,\phi) = \Phi_1$ and $V_o(-\mu_2,\eta,\phi) = \Phi_2$. Hence, considering the terms independent of $\cos\phi$,

$$\Phi_1 = -\frac{E c \sinh\mu_1 \cos\psi}{(\cosh\mu_1 - \cos\eta)}$$

$$+ (\cosh\mu_1 - \cos\eta)^{1/2} \sum_{n=0}^{\infty} \left[A_n e^{(n+\frac{1}{2})\mu_1} + B_n e^{-(n+\frac{1}{2})\mu_1}\right] P_n(\cos\eta) \qquad (B1)$$

$$\Phi_2 = \frac{E c \sinh\mu_2 \cos\psi}{(\cosh\mu_2 - \cos\eta)}$$

$$+ (\cosh\mu_2 - \cos\eta)^{1/2} \sum_{n=0}^{\infty} \left[A_n e^{-(n+\frac{1}{2})\mu_2} + B_n e^{(n+\frac{1}{2})\mu_2}\right] P_n(\cos\eta) \qquad (B2)$$

Divide equation (B1) by $(\cosh\mu_1 - \cos\eta)^{1/2}$ and equation (B2) by $(\cosh\mu_2 - \cos\eta)^{1/2}$ and use equations (A5) and (A6). Then equations (B1) and (B2) become

$$\sqrt{2}\Phi_1 \sum_{n=0}^{\infty} e^{-(n+\frac{1}{2})\mu_1} P_n(\cos\eta) = -2\sqrt{2} E c \cos\psi \sum_{n=0}^{\infty} (n+\tfrac{1}{2}) e^{-(n+\frac{1}{2})\mu_1} P_n(\cos\eta)$$

$$+ \sum_{n=0}^{\infty} \left[A_n e^{(n+\frac{1}{2})\mu_1} + B_n e^{-(n+\frac{1}{2})\mu_1} \right] P_n(\cos\eta) \qquad \text{(B3)}$$

$$\sqrt{2}\Phi_2 \sum_{n=0}^{\infty} e^{-(n+\frac{1}{2})\mu_2} P_n(\cos\eta) = 2\sqrt{2}E\cos\psi \sum_{n=0}^{\infty} (n+\tfrac{1}{2}) e^{-(n+\frac{1}{2})\mu_2} P_n(\cos\eta)$$

$$+ \sum_{n=0}^{\infty} \left[A_n e^{-(n+\frac{1}{2})\mu_2} + B_n e^{(n+\frac{1}{2})\mu_2} \right] P_n(\cos\eta) \qquad \text{(B4)}$$

It follows that

$$A_n e^{(n+\frac{1}{2})\mu_1} + B_n e^{-(n+\frac{1}{2})\mu_1} = 2\sqrt{2}(n+\tfrac{1}{2})E\cos\psi e^{-(n+\frac{1}{2})\mu_1} + \sqrt{2}\Phi_1 e^{-(n+\frac{1}{2})\mu_1} \qquad \text{(B5)}$$

$$A_n e^{-(n+\frac{1}{2})\mu_2} + B_n e^{(n+\frac{1}{2})\mu_2} = 2\sqrt{2}(n+\tfrac{1}{2})E\cos\psi e^{-(n+\frac{1}{2})\mu_2} + \sqrt{2}\Phi_2 e^{-(n+\frac{1}{2})\mu_2} \qquad \text{(B6)}$$

Solving for A_n and B_n,

$$A_n = \frac{\sqrt{2}E\cos\psi(2n+1)\left[e^{(2n+1)\mu_2}+1\right] + \sqrt{2}\left[\Phi_1 e^{(2n+1)\mu_2} - \Phi_2\right]}{e^{(2n+1)(\mu_1+\mu_2)} - 1} \qquad \text{(B7)}$$

$$B_n = \frac{-\sqrt{2}E\cos\psi(2n+1)\left[e^{(2n+1)\mu_1}+1\right] + \sqrt{2}\left[\Phi_2 e^{(2n+1)\mu_1} - \Phi_1\right]}{e^{(2n+1)(\mu_1+\mu_2)} - 1} \qquad \text{(B8)}$$

We now consider the terms in equation (II.11) for V_o which are proportional to $\cos\phi$. Putting $\mu = \mu_1$ and $\mu = -\mu_2$, and noting that Φ_1 and Φ_2 are independent of ϕ, we obtain the boundary conditions

$$0 = -\frac{E\sin\eta\sin\psi}{(\cosh\mu_1-\cos\eta)} + \sum_{n=1}^{\infty}\left[C_n e^{(n+\frac{1}{2})\mu_1} + D_n e^{-(n+\frac{1}{2})\mu_1}\right]P_n^1(\cos\eta) \tag{B9}$$

$$0 = -\frac{E\sin\eta\sin\psi}{(\cosh\mu_2-\cos\eta)} + \sum_{n=1}^{\infty}\left[C_n e^{-(n+\frac{1}{2})\mu_2} + D_n e^{(n+\frac{1}{2})\mu_2}\right]P_n^1(\cos\eta) \tag{B10}$$

Dividing again by $(\cosh\mu_1-\cos\eta)^{1/2}$ and $(\cosh\mu_2-\cos\eta)^{1/2}$ and using the relation (the derivative of equation (A5) with respect to η)

$$\frac{\sin\eta}{(\cosh\mu-\cos\eta)^{3/2}} = 2\sqrt{2}\sum_{n=1}^{\infty} e^{-(n+\frac{1}{2})\mu} P_n^1(\cos\eta) \tag{B11}$$

where $\mu = \mu_1$ or μ_2, we obtain the two equations determining C_n and D_n,

$$C_n e^{(n+\frac{1}{2})\mu_1} + D_n e^{-(n+\frac{1}{2})\mu_1} = 2\sqrt{2}E\sin\psi e^{-(n+\frac{1}{2})\mu_1} \tag{B12}$$

$$C_n e^{-(n+\frac{1}{2})\mu_2} + D_n e^{(n+\frac{1}{2})\mu_2} = 2\sqrt{2}E\sin\psi e^{-(n+\frac{1}{2})\mu_2} \tag{B13}$$

It follows that

$$C_n = \frac{2\sqrt{2}E\sin\psi\left[e^{(2n+1)\mu_2}-1\right]}{e^{(2n+1)(\mu_1+\mu_2)}-1} \tag{B14}$$

$$D_n = \frac{2\sqrt{2}E\sin\psi\left[e^{(2n+1)\mu_1}-1\right]}{e^{(2n+1)(\mu_1+\mu_2)}-1} \tag{B15}$$

Derivation of Equations (8) and (10)

Consider the different contributions to $(\partial V_o/\partial\mu)_{\mu=-\mu_2}$ from the expression (II.11) for $V_o(\mu,\eta,\phi)$ starting with the terms independent of $\cos\phi$. We make use of the formula which replaces equation (A6) when $\mu < 0$, namely

$$\frac{\sinh\mu}{(\cosh\mu-\cos\eta)^{3/2}} = -2\sqrt{2}\sum_{n=0}^{\infty}(n+\tfrac{1}{2})e^{(n+\frac{1}{2})\mu}P_n(\cos\eta) \qquad \text{(B16)}$$

It follows that one contribution to $(\partial V_o/\partial\mu)_{\mu=-\mu_2}$ is given by

$$\frac{\partial}{\partial\mu}\left[\frac{-E c\sinh\mu\cos\psi}{\cosh\mu-\cos\eta}\right]_{\mu=-\mu_2} = -\frac{\sqrt{2}Ec\sinh\mu_2\cos\psi}{(\cosh\mu_2-\cos\eta)^{1/2}}\sum_{n=0}^{\infty}(n+\tfrac{1}{2})e^{-(n+\frac{1}{2})\mu_2}P_n(\cos\eta)$$

$$+ 2\sqrt{2}Ec\cos\psi(\cosh\mu_2-\cos\eta)^{1/2}\sum_{n=0}^{\infty}(n+\tfrac{1}{2})^2e^{-(n+\frac{1}{2})\mu_2}P_n(\cos\eta) \qquad \text{(B17)}$$

If we now use equation (B6), the right-hand side of equation (B17) becomes

$$\tfrac{1}{2}(\cosh\mu_2-\cos\eta)^{1/2}\sum_{n=0}^{\infty}\left[A_n e^{-(n+\frac{1}{2})\mu_2} + B_n e^{(n+\frac{1}{2})\mu_2} - \sqrt{2}\Phi_2 e^{-(n+\frac{1}{2})\mu_2}\right]$$

$$\times\left[\frac{\sinh\mu_2}{(\cosh\mu_2-\cos\eta)} - 2(n+\tfrac{1}{2})\right]P_n(\cos\eta) \qquad \text{(B18)}$$

The other contribution which is independent of $\cos\phi$ is

$$\frac{\partial}{\partial\mu}\left[(\cosh\mu-\cos\eta)^{1/2}\sum_{n=0}^{\infty}\left[A_n e^{(n+\frac{1}{2})\mu}+B_n e^{-(n+\frac{1}{2})\mu}\right]P_n(\cos\eta)\right]_{\mu=-\mu_2}$$

$$-\tfrac{1}{2}\frac{\sinh\mu_2}{(\cosh\mu_2-\cos\eta)^{1/2}}\sum_{n=0}^{\infty}\left[A_n e^{-(n+\frac{1}{2})\mu}+\tfrac{1}{2}B_n e^{(n+\frac{1}{2})\mu_2}\right]P_n(\cos\eta)$$

$$+(\cosh\mu_2-\cos\eta)^{1/2}\sum_{n=0}^{\infty}(n+\tfrac{1}{2})\left[A_n e^{-(n+\frac{1}{2})\mu_2}-B_n e^{(n+\frac{1}{2})\mu_2}\right]P_n(\cos\eta) \qquad \text{(B19)}$$

The sum of equations (B18) and (B19) is

$$\frac{1}{\sqrt{2}}\Phi_2(\cosh\mu_2-\cos\eta)^{1/2}\sum_{n=0}^{\infty}\left[\frac{-\sinh\mu_2}{(\cosh\mu_2-\cos\eta)}+2(n+\tfrac{1}{2})\right]e^{-(n+\frac{1}{2})\mu_2}P_n(\cos\eta)$$

$$-2(\cosh\mu_2-\cos\eta)^{1/2}\sum_{n=0}^{\infty}(n+\tfrac{1}{2})B_n e^{(n+\frac{1}{2})\mu_2}P_n(\cos\eta) \qquad \text{(B20)}$$

On using equations (A5) and (A6) evaluated at $\mu = \mu_2$, it is readily verified that the first and second terms in equation (B20) cancel, so that the part of $(\partial V_o/\partial\mu)_{\mu=-\mu_2}$ which is independent of $\cos\phi$ is the third term only in equation (B20).

We now consider the term in V_o which is proportional to $\cos\phi$. Using the equation (B11) for $\mu < 0$, namely

$$\frac{\sin\eta}{(\cosh\mu-\cos\eta)^{3/2}} = 2\sqrt{2}\sum_{n=1}^{\infty}e^{(n+\frac{1}{2})\mu}P_n^1(\cos\eta) \qquad \text{(B21)}$$

we write, for $\mu < 0$

$$\frac{\sin\eta}{\cosh\mu-\cos\eta} - \frac{\sin\eta(\cosh\mu-\cos\eta)^{1/2}}{(\cosh\mu-\cos\eta)^{3/2}}$$

$$- 2\sqrt{2}(\cosh\mu-\cos\eta)^{1/2} \sum_{n-1}^{\infty} (n+\tfrac{1}{2})e^{(n+\frac{1}{2})\mu} P_n^1(\cos\eta) \qquad \text{(B22)}$$

Hence

$$\frac{\partial}{\partial\mu}\left[-\frac{E c\sin\psi\sin\eta}{\cosh\mu-\cos\eta}\right]_{\mu=-\mu_2} - \sqrt{2}Ec\frac{\sin\psi\sinh\mu_2}{(\cosh\mu_2-\cos\eta)^{1/2}} \sum_{n-1}^{\infty} e^{-(n+\frac{1}{2})\mu_2} P_n^1(\cos\eta)$$

$$- 2\sqrt{2}Ec\sin\psi(\cosh\mu_2-\cos\eta)^{1/2} \sum_{n-1}^{\infty} (n+\tfrac{1}{2})e^{-(n+\frac{1}{2})\mu_2} P_n^1(\cos\eta) \qquad \text{(B23)}$$

On making use of equation (B13), we can write equation (B23) as

$$\frac{\partial}{\partial\mu}\left[-\frac{E c\sin\psi\sin\eta}{\cosh\mu-\cos\eta}\right]_{\mu=-\mu_2}$$

$$- \frac{\sinh\mu_2}{2(\cosh\mu_2-\cos\eta)^{1/2}} \sum_{n-1}^{\infty} \left[C_n e^{-(n+\frac{1}{2})\mu_2} + D_n e^{(n+\frac{1}{2})\mu_2}\right] P_n^1(\cos\eta)$$

$$- (\cosh\mu_2-\cos\eta)^{1/2} \sum_{n-1}^{\infty} \left[C_n e^{-(n+\frac{1}{2})\mu_2} + D_n e^{(n+\frac{1}{2})\mu_2}\right] P_n^1(\cos\eta) \qquad \text{(B24)}$$

The other term in V_o proportional to $\cos\phi$ contributes the corresponding expression

$$\frac{\partial}{\partial\mu}\left[(\cosh\mu-\cos\eta)^{1/2} \sum_{n-1}^{\infty} \left[C_n e^{(n+\frac{1}{2})\mu} + D_n e^{-(n+\frac{1}{2})\mu}\right] P_n^1(\cos\eta)\right]_{\mu=-\mu_2}$$

$$- \frac{-\sinh\mu_2}{(\cosh\mu_2-\cos\eta)^{1/2}} \sum_{n=1}^{\infty} \left[C_n e^{-(n+\frac{1}{2})\mu_2} + D_n e^{(n+\frac{1}{2})\mu_2} \right] P_n^1(\cos\eta)$$

$$- (\cosh\mu_2-\cos\eta)^{1/2} \sum_{n=1}^{\infty} (n+\tfrac{1}{2}) \left[C_n e^{-(n+\frac{1}{2})\mu_2} - D_n e^{(n+\frac{1}{2})\mu_2} \right] P_n^1(\cos\eta) \qquad \text{(B25)}$$

Adding equations (B24) and (B25), the term in $\cos\phi$ in V_o contributes to $(\partial V_o/\partial\mu)_{\mu=-\mu_2}$

$$-2(\cosh\mu_2-\cos\eta)^{1/2} \sum_{n=1}^{\infty} (n+\tfrac{1}{2}) D_n e^{(n+\frac{1}{2})\mu_2} P_n^1(\cos\eta) \qquad \text{(B26)}$$

The results derived in equations (B20) and (B26) therefore yield

$$\left[\frac{\partial V_o(\mu,\eta,\phi)}{\partial\mu}\right]_{\mu=-\mu_2} = -(\cosh\mu_2-\cos\eta)^{1/2} \left[\sum_{n=0}^{\infty} (2n+1) B_n e^{(n+\frac{1}{2})\mu_2} P_n(\cos\eta) \right.$$

$$\left. + \cos\phi \sum_{n=1}^{\infty} (2n+1) D_n e^{(n+\frac{1}{2})\mu_2} P_n^1(\cos\eta) \right] \qquad \text{(B27)}$$

which is identical with that stated by Davis [13].

Again begin with the terms independent of $\cos\phi$ in V_o. Substituting equations (A6) and (B5),

$$\frac{\partial}{\partial\mu}\left[\frac{-E c \sinh\mu \cos\psi}{\cosh\mu-\cos\eta}\right]_{\mu=\mu_1} = -\frac{\sqrt{2}E c \sinh\mu_1 \cos\psi}{(\cosh\mu_1-\cos\eta)^{1/2}} \sum_{n=0}^{\infty} (n+\tfrac{1}{2}) e^{-(n+\frac{1}{2})\mu_1} P_n(\cos\eta)$$

$$+ 2\sqrt{2}Ec\cos\phi(\cosh\mu_1-\cos\eta)^{1/2}\sum_{n=0}^{\infty}(n+\tfrac{1}{2})^2 e^{-(n+\frac{1}{2})\mu_1}P_n(\cos\eta)$$

$$= -\tfrac{1}{2}(\cosh\mu_2-\cos\eta)^{1/2}\sum_{n=0}^{\infty}\left[A_n e^{(n+\frac{1}{2})\mu_1} + B_n e^{-(n+\frac{1}{2})\mu_1} - \sqrt{2}\Phi_1 e^{-(n+\frac{1}{2})\mu_1}\right]$$

$$\times\left[(2n+1) - \frac{\sinh\mu_1}{(\cosh\mu_1-\cos\eta)}\right]P_n(\cos\eta) \qquad \text{(B28)}$$

The other contribution to $(\partial V_o/\partial\mu)$ at $\mu = \mu_1$ which is independent of $\cos\phi$ is

$$\frac{\partial}{\partial\mu}\left[(\cosh\mu-\cos\eta)^{1/2}\sum_{n=0}^{\infty}\left[A_n e^{(n+\frac{1}{2})\mu} + B_n e^{-(n+\frac{1}{2})\mu}\right]P_n(\cos\eta)\right]_{\mu=\mu_1}$$

$$= -\frac{\sinh\mu_1}{2(\cosh\mu_2-\cos\eta)^{1/2}}\sum_{n=0}^{\infty}\left[A_n e^{(n+\frac{1}{2})\mu_1} + B_n e^{-(n+\frac{1}{2})\mu_1}\right]P_n(\cos\eta)$$

$$+ (\cosh\mu_1-\cos\eta)^{1/2}\sum_{n=0}^{\infty}(n+\tfrac{1}{2})\left[A_n e^{(n+\frac{1}{2})\mu_1} - B_n e^{-(n+\frac{1}{2})\mu_1}\right]P_n(\cos\eta) \qquad \text{(B29)}$$

The sum of equations (B28) and (B30) is

$$-\frac{\Phi_1}{\sqrt{2}}(\cosh\mu_1-\cos\eta)^{1/2}\sum_{n=0}^{\infty}e^{-(n+\frac{1}{2})\mu_1}P_n(\cos\eta)\left[\frac{\sinh\mu_1}{(\cosh\mu_1-\cos\eta)} - (2n+1)\right]$$

$$+ (\cosh\mu_1-\cos\eta)^{1/2}\sum_{n=0}^{\infty}(2n+1)A_n e^{(n+\frac{1}{2})\mu_1}P_n(\cos\eta) \qquad \text{(B30)}$$

Just as in equation (B20), the first and second terms in equation (B30) cancel, and so the part which is independent of $\cos\phi$ is the third term only in equation (B30).

Again we consider the term in V_o which is proportional to $\cos\phi$. Using equation (B11), we write for $\mu > 0$

$$\frac{\sin\eta}{\cosh\mu-\cos\eta} = 2\sqrt{2}(\cosh\mu-\cos\eta)^{1/2} \sum_{n=1}^{\infty} e^{-(n+\frac{1}{2})\mu} P_n^1(\cos\eta) \tag{B31}$$

Making use of equation (B12), it follows that

$$\frac{\partial}{\partial\mu}\left[\frac{-E c \sinh\mu\cos\phi}{\cosh\mu-\cos\eta}\right]_{\mu=\mu_1} = -\frac{\sqrt{2}Ec\sinh\mu_1\sin\phi}{(\cosh\mu_1-\cos\eta)^{1/2}} \sum_{n=1}^{\infty} e^{-(n+\frac{1}{2})\mu_1} P_n^1(\cos\eta)$$

$$+ \sqrt{2}Ec\sin\phi(\cosh\mu_1-\cos\eta)^{1/2} \sum_{n=1}^{\infty} (2n+1)e^{-(n+\frac{1}{2})\mu_1} P_n^1(\cos\eta)$$

$$- \frac{1}{2}(\cosh\mu_1-\cos\eta)^{1/2} \sum_{n=1}^{\infty} \left[C_n e^{(n+\frac{1}{2})\mu_1} + D_n e^{-(n+\frac{1}{2})\mu_1}\right] P_n^1(\cos\eta)$$

$$\times \left[(2n+1) - \frac{\sinh\mu_1}{(\cosh\mu_1-\cos\eta)}\right] \tag{B32}$$

The other term in V_o which is proportional to $\cos\phi$ contributes

$$\frac{\partial}{\partial\mu}\left((\cosh\mu-\cos\eta)^{1/2} \sum_{n=1}^{\infty} \left[C_n e^{(n+\frac{1}{2})\mu} + D_n e^{-(n+\frac{1}{2})\mu}\right] P_n^1(\cos\eta)\right)_{\mu=\mu_1}$$

$$- \frac{\sinh\mu_1}{2(\cosh\mu_1-\cos\eta)^{1/2}} \sum_{n=1}^{\infty} \left[C_n e^{(n+\frac{1}{2})\mu_1} + D_n e^{-(n+\frac{1}{2})\mu_1} \right] P_n^1(\cos\eta)$$

$$+ (\cosh\mu_1-\cos\eta)^{1/2} \sum_{n=1}^{\infty} (n+\tfrac{1}{2}) \left[C_n e^{(n+\frac{1}{2})\mu_1} - D_n e^{-(n+\frac{1}{2})\mu_1} \right] P_n^1(\cos\eta) \qquad \text{(B33)}$$

Adding equations (B32) and (B33) yields

$$(\cosh\mu_1-\cos\eta)^{1/2} \sum_{n=1}^{\infty} (2n+1) C_n e^{(n+\frac{1}{2})\mu_1} P_n^1(\cos\eta) \qquad \text{(B34)}$$

From equations (B20) and (B33), we obtain the desired result

$$\left[\frac{\partial V_o(\mu,\eta,\phi)}{\partial\mu} \right]_{\mu=\mu_1} = (\cosh\mu_1-\cos\eta)^{1/2} \left[\sum_{n=0}^{\infty} (2n+1) A_n e^{(n+\frac{1}{2})\mu_1} P_n(\cos\eta) \right.$$

$$\left. + \cos\phi \sum_{n=1}^{\infty} (2n+1) C_n e^{(n+\frac{1}{2})\mu_1} P_n^1(\cos\eta) \right] \qquad \text{(B35)}$$

Equation (B35) is, again, the same as that quoted by Davis [13].

IV. INTERACTION OF TWO UNCHARGED CONDUCTING SPHERES IN AN ELECTRIC FIELD

Equal Spheres Aligned Parallel to the Field

We wish to find the electrostatic free energy associated with two equal, uncharged, perfectly conducting spheres in the presence of an applied electric field. The electrostatic interaction energy is the change in the quantity -Em/2, where m is the total dipole moment to be attributed to the two spheres when they are brought from infinite separation to a specified separation. For simplicity, it is assumed that the centres of the two spheres lie on the z-axis, which is parallel to the direction of the applied field so that the angle $\phi = 0$. Origin of coordinates is taken midway between the two spheres. The Cartesian coordinates can be expressed in terms of the bispherical coordinates as (Ref. [14], p. 110)

$$x^2 + y^2 + (z - c\coth\mu)^2 = \frac{c^2}{\sinh^2\mu} \tag{1}$$

$$x^2 + y^2 + z^2 - 2c(x^2+y^2)^{1/2}\cot\eta = c^2 \tag{2}$$

$$\tan\phi = y/x \tag{3}$$

where μ = constant is a sphere, η = constant is a spindle-shaped surface, and ϕ = constant is a half-plane.

Our objective is to determine m by expressing the potential $V_o(\mu,\eta,\phi)$ in the form

$$V_o(\mu,\eta,\phi) = -Ez + \frac{mz}{\epsilon_o r^3} + \ldots \tag{4}$$

where ϵ_o is the dielectric constant of the oil and $r = (x^2+y^2+z^2)^{1/2}$ is the distance from the origin. The potential for two spheres with arbitrary charges, radii and aligned at an arbitrary angle to the field is given by equation (II.11). For $\mu_1 = \mu_2 = \mu_0$, $\psi = 0$, and zero charge on the spheres, the constants A_n and B_n in this equation become

$$A_n = -B_n = \sqrt{2}Ec\left[(2n+1) - \frac{\alpha_0}{\beta_1+\beta_2}\right]\left[e^{(2n+1)\mu_0} - 1\right]^{-1} \tag{5}$$

since the potentials at the spheres have been shown to be (see equation (III.57)

$$\Phi_1 = -\Phi_2 = \frac{-\alpha_0 Ec}{\beta_1 + \beta_2} \tag{6}$$

where

$$\alpha_0 = \sum_{n=0}^{\infty} (n+\tfrac{1}{2})\frac{e^{-(n+\frac{1}{2})\mu_0}}{\sinh(n+\frac{1}{2})\mu_0}, \quad \beta_1 + \beta_2 = \tfrac{1}{2}\sum_{n=0}^{\infty}\frac{e^{-(n+\frac{1}{2})\mu_0}}{\sinh(n+\frac{1}{2})\mu_0} \tag{7}$$

Furthermore, the two remaining constants in equation (II.11) are

$$C_n = D_n = 0 \tag{8}$$

Thus, the potential $V_o(\mu,\eta,\phi)$ is given by

$$V_o(\mu,\eta,\phi) = -Ez + 2(\cosh\mu-\cos\eta)^{1/2} \sum_{n=0}^{\infty} A_n \sinh(n+\tfrac{1}{2})\mu \; P_n(\cos\eta) \qquad (9)$$

Expanding the terms in the summation in powers of 1/r (Appendix C), we obtain V_o to order $1/r^3$

$$V_o(\mu,\eta,\phi) = -Ez + 2\sqrt{2}\frac{zc^2}{r^3} \sum_{n=0}^{\infty} (2n+1)A_n \qquad (10)$$

and on comparing with equation (4), the dipole moment is given by

$$m = 2\sqrt{2}\epsilon_o c^2 \sum_{n=0}^{\infty} (2n+1)A_n \qquad (11)$$

It is seen that m is proportional to E. When the two spheres are at infinite separation, the dipole moment of each sphere equals $m_0 = \alpha\epsilon_o E = a^3\epsilon_o E$ (ϵ_o is the dielectric constant of the suspending oil medium), since the polarizability of a perfectly conducting sphere of radius a equals a^3 in vacuo. Thus, the electrostatic interaction energy of the two spheres is

$$-\tfrac{1}{2}(m-2m_0)E = -\tfrac{1}{2}(m-2a^3\epsilon_o E)E \;, \quad a = \frac{c}{\sinh\mu_0} \qquad (12)$$

It is possible to arrive at the expression -mE/2 for the electrostatic energy without using an unusual model of Landau and Lifshitz [23], which we describe in Chapter VI. The induced dipole of a set of bound charges due to a uniform external electric field E is $\mu = \alpha E$, where α is the polarizability of the set of charges. The energy of the induced dipole is

$$-\int_0^E \mu dE = -\alpha\int_0^E EdE = -\frac{1}{2}\alpha E^2 = -\frac{1}{2}\mu E \qquad (13)$$

Unequal Spheres at Arbitrary Angle

We generalise the method described above for determining the electrostatic interaction energy associated with two uncharged perfectly conducting spheres in the presence of an applied electric field. The radii of the spheres are unequal and their centres lie on a line at an arbitrary angle ψ to the direction of the applied field. In Cartesian coordinates x,y,z, the two centres are situated on the z-axis and the direction of the electric field, $\underset{\sim}{E}$, is in the x-z plane. At large distances from the two spheres the potential $V_o(\mu,\eta,\phi)$, where μ,η,ϕ are the bipolar coordinates, takes the form

$$V_o(\mu,\eta,\phi) = -Ez\cos\psi - Ex\sin\psi + \frac{m_z z}{\epsilon_o r^3} + \frac{m_x x}{\epsilon_o r^3} + \ldots \qquad (14)$$

where m_x and m_z are the x- and z-components of $\underset{\sim}{m}$, the total dipole moment induced by the electric field on the two uncharged spheres. ϵ_o is the dielectric constant of the suspending oil medium and r is the distance from the coordinate origin which is situated between the two spheres. The required electrostatic energy is

$$-\frac{1}{2}\underset{\sim}{E}\cdot\underset{\sim}{m} = -\frac{1}{2}(E_x m_x + E_z m_z) = -\frac{1}{2}E(m_x\sin\psi + m_z\cos\psi) \qquad (15)$$

The objective is to find m_x and m_z.

The total potential is given by

$$V_o(\mu,\eta,\phi) = -Ez\cos\psi - Ex\sin\psi$$

$$+ (\cosh\mu-\cos\eta)^{1/2} \sum_{n=0}^{\infty} \left[A_n e^{(n+\frac{1}{2})\mu} + B_n e^{-(n+\frac{1}{2})\mu}\right] P_n(\cos\eta)$$

$$+ \cos\phi \sum_{n=0}^{\infty} \left[C_n e^{(n+\frac{1}{2})\mu} + D_n e^{-(n+\frac{1}{2})\mu}\right] P_n^1(\cos\eta) \qquad (16)$$

We need to eliminate $\cos\phi$ from the last term in equation (16). Using bipolar coordinates (equation (II.1)),

$$\cos\phi = \frac{x(\cosh\mu-\cos\eta)}{c\sin\eta} \qquad (17)$$

and writing

$$P_n^1(\cos\eta) = \sin\eta \frac{dP_n(\cos\eta)}{d(\cos\eta)} \qquad (18)$$

we can express the $\cos\phi$ term in equation (16) as

$$\frac{x}{c}(\cosh\mu-\cos\eta)^{3/2} \sum_{n=0}^{\infty} \left[C_n e^{(n+\frac{1}{2})\mu} + D_n e^{-(n+\frac{1}{2})\mu}\right] \frac{dP_n(\cos\eta)}{d(\cos\eta)} \qquad (19)$$

We have shown that the potentials of the two uncharged spheres are (equation (III.56)

$$\Phi_1 = \lambda_3 Ec\cos\psi \ , \quad \Phi_2 = r_3 Ec\cos\psi \qquad (20)$$

where

$$\lambda_3 = \frac{(\beta_1\beta_0 - \alpha_0\beta_2)}{(\alpha_1\beta_2 - \beta_1^2)}, \quad \tau_3 = \frac{(\alpha_1\beta_0 - \alpha_0\beta_1)}{(\alpha_1\beta_2 - \beta_1^2)} \tag{21}$$

α_0, α_1, β_0, β_1 and β_2 are infinite series in terms of the bipolar coordinates μ_1 and μ_2 which define the two spheres (equations (III.32) - (III.34), (III.37), and (III.38)). It follows from equations (III.3) - (III.6) that the coefficients A_n, B_n, C_n, and D_n in equations (16) and (19) become

$$A_n = \sqrt{2}E\cos\psi \frac{(2n+1)\left[e^{(2n+1)\mu_2} + 1\right] + \lambda_3 e^{(2n+1)\mu_2} - \tau_3}{e^{(2n+1)(\mu_1+\mu_2)} - 1} \tag{22}$$

$$B_n = \sqrt{2}E\cos\psi \frac{-(2n+1)\left[e^{(2n+1)\mu_1} + 1\right] + \tau_3 e^{(2n+1)\mu_1} - \lambda_3}{e^{(2n+1)(\mu_1+\mu_2)} - 1} \tag{23}$$

$$C_n = 2\sqrt{2}E\sin\psi \frac{e^{(2n+1)\mu_2} - 1}{e^{(2n+1)(\mu_1+\mu_2)} - 1} \tag{24}$$

$$D_n = 2\sqrt{2}E\sin\psi \frac{e^{(2n+1)\mu_1} - 1}{e^{(2n+1)(\mu_1+\mu_2)} - 1} \tag{25}$$

Using the expansions given by equations (C.1) - (C.12) in Appendix C, we can expand $V_o(\mu,\eta,\phi)$ in inverse powers of r to obtain

$$V_o(\mu,\eta,\phi) = -Ez\cos\psi - Ex\sin\psi + \frac{\sqrt{2}c}{r}\sum_{n=0}^{\infty}(A_n+B_n)$$

$$- \frac{1}{\sqrt{2}}\frac{c^3}{r^3}\sum_{n=0}^{\infty}(A_n+B_n) + \frac{2\sqrt{2}c^2z}{r^3}\sum_{n=0}^{\infty}(n+\tfrac{1}{2})(A_n-B_n)$$

$$- \frac{2\sqrt{2}c^2z}{r^3}\sum_{n=0}^{\infty}(A_n+B_n)\left[\frac{dP_n(u)}{du}\right]_{u=1}$$

$$+ \frac{2\sqrt{2}c^2x}{r^3}\sum_{n=0}^{\infty}(C_n+D_n)\left[\frac{dP_n(u)}{du}\right]_{u=1} + \ldots \qquad (26)$$

The potential function in equation (16), with coefficients A_n, B_n, C_n, D_n given by equations (22) - (24), describes two uncharged spheres in the presence of an applied electric field at angle ψ to the line of centres. The first two terms in equation (16) account for the applied field and the rest the potential distribution due to the two uncharged spheres in which a displacement of charge due to the field has taken place. We notice that the coefficients A_n, B_n, C_n, D_n in equations (22) - (24) are proportional to E, there being no terms proportional to charges $Q_1 = Q_2 = 0$. Assuming that our multipole expansion in powers of $\frac{1}{r}$ is correct, it follows that in equation (26) terms in $\frac{1}{r}$ should be absent, i.e.

$$\sum_{n=0}^{\infty}(A_n+B_n) = 0 \qquad (27)$$

Indeed if $\mu_1 = \mu_2$, $A_n = -B_n$ and this condition is satisfied, but the relation (27) is not easily shown when $\mu_1 \neq \mu_2$. We refer to Appendix C for a proof.

Comparison of equation (1) with equation (26) yields the induced dipole moments m_z and m_x:

$$m_z = 2\sqrt{2}c^2\epsilon_o \sum_{n=0}^{\infty} \left[(n+\tfrac{1}{2})(A_n - B_n) - c(A_n + B_n) \left[\frac{dP_n(u)}{du} \right]_{u=1} \right] \tag{28}$$

$$m_x = 2\sqrt{2}c^2\epsilon_o \sum_{n=0}^{\infty} (C_n + D_n) \left[\frac{dP_n(u)}{du} \right]_{u=1} \tag{29}$$

Observe that m_z is proportional to $E\cos\psi$ and m_x to $E\sin\psi$ so that the energy in equation (15) has the expected form

$$U = -E^2(f\cos^2\psi + g\sin^2\psi) \tag{30}$$

Quadrupole and higher order multipole terms do not contribute to the electrostatic energy for a uniform applied field.

We are now able to determine m_z and m_x in equation (14) for two unequal, uncharged conducting spheres at arbitrary angle relative to the applied field in terms of Davis's functions $S_m(\xi)$, which are defined by [13]

$$S_m(\xi) = \sum_{n=0}^{\infty} (2n+1)^m \frac{e^{(2n+1)\xi}}{e^{(2n+1)(\mu_1+\mu_2)} - 1} \tag{31}$$

Noting that

$$\left[\frac{dP_n(u)}{du} \right]_{u=1} = \tfrac{1}{2}(n^2+n) = \tfrac{1}{8}[(2n+1)^2 - 1] \tag{32}$$

and making use of equation (27), we can write

$$m_z = \sqrt{2}c^2\epsilon_o \sum_{n=0}^{\infty} \left[(2n+1)(A_n-B_n) - \tfrac{1}{4}(2n+1)^2(A_n+B_n)\right] \tag{33}$$

On substituting equations (22) and (23) for A_n and B_n and using equation (31), equation (33) becomes

$$m_z = 2Ec^3\cos\psi\epsilon_o\Big[S_2(\mu_2)+S_2(\mu_1)+2S_2(0)+\lambda_3 S_1(\mu_2)-\tau_3 S_1(\mu_1)$$

$$+ (\lambda_3-\tau_3)S_1(0) - \tfrac{1}{4}\left[S_3(\mu_2)-S_3(\mu_1)+\lambda_3 S_1(\mu_2)-\tau_3 S_1(\mu_1)-(\lambda_3+\tau_3)S_2(0)\right]\Big] \tag{34}$$

From equation (29),

$$m_x = \frac{\sqrt{2}}{4}c^2\epsilon_o \sum_{n=0}^{\infty} (C_n+D_n)[(2n+1)^2-1]$$

$$- Ec^3\sin\psi\epsilon_o\left[S_2(\mu_2)+S_2(\mu_1)-2S_2(0)-S_0(\mu_2)-S_0(\mu_1)+2S_0(0)\right] \tag{35}$$

No marked simplification due to cancellation results from substitution of equations (C.20) and (C.21) for λ_3 and τ_3. For the special case of equal radii, when we put $\mu_1 = \mu_2 = \mu_0$,

$$\tau_3 = -\lambda_3 = -\frac{\alpha_0}{\beta_1+\beta_2} = -\frac{\hat{S}_1(\mu_0)+\hat{S}_1(0)}{\hat{S}_0(\mu_0)+\hat{S}_0(0)} \tag{36}$$

$$m_z = 4Ec^3\cos\psi\epsilon_o\left[\hat{S}_2(\mu_0)+\hat{S}_2(0)+\lambda_3\left[\hat{S}_1(\mu_0)+\hat{S}_1(0)\right]\right] \tag{37}$$

$$m_x = 2Ec^3 \sin\psi \epsilon_o [\hat{S}_2(\mu_0) - \hat{S}_2(0) - \hat{S}_0(\mu_0) + \hat{S}_0(0)] \tag{38}$$

where α_0, β_0, and β_1 are defined in equation (III.47) and $\hat{S}_m(\xi)$ is obtained from $S_m(\xi)$ by putting $\mu_1 = \mu_2 = \mu_0$, namely

$$\hat{S}_m(\xi) = \sum_{n=0}^{\infty} \frac{(2n+1)^m e^{(2n+1)\xi}}{e^{2(2n+1)\mu_0} - 1} \tag{39}$$

For two unequal, uncharged spheres, the interaction energy depends on five parameters: the two radii R_1 and R_2, the electric field E, the distance between the centres of the spheres and the angle of orientation ψ. In terms of bipolar coordinates, this interaction energy is expressed in terms of the parameters μ_1, μ_2, c, E, and ψ. Having obtained the interaction energy, we would like to define in terms of bipolar coordinates the component of the force between the spheres along the line of centres, i.e. in the z-direction. We should also consider the force on either sphere in the x-direction, where the direction of the electric field is in the x-z plane at angle ψ to the z-axis. Davis [13] determined these force components by integrating the Maxwell electrostatic stress over the surface of a sphere. Here we need to take appropriate derivatives of the interaction energy. We repeat the relations, already presented in Chapter III, between the dimensionless parameters μ_1, μ_2 and the length c, and the three lengths, sphere radii R_1, R_2, and the distance between the centres of the spheres. The distances from the origin of bipolar coordinates to the centres of the two spheres are

$$D_1 = c\coth\mu_1 - R_1\cosh\mu_1 \ , \quad D_2 = c\coth\mu_2 - R_2\cosh\mu_2 \tag{40}$$

The force component in the z-direction on sphere 1 can be obtained by differentiating with respect to D_1, keeping D_2 fixed. Since R_1 and R_2 are constant, it follows from equation (40) that we differentiate with respect to μ_1 only, i.e. at fixed μ_2, c, E, and ψ. The force in the z-direction on sphere 1 is, therefore,

$$\frac{\partial U}{\partial D_1} = \left[\frac{\partial U}{\partial \mu_1}\right]_{\mu_2,c,E,\psi}\left[\frac{\partial \mu_1}{\partial D_1}\right]_{R_1} \tag{41}$$

where

$$\mu_1 = \ln\left[\frac{D_1 + (D_1^2 - R_1^2)^{1/2}}{R_1}\right] \tag{42}$$

To obtain the force component in the x-direction, imagine a small displacement dx of the centre of sphere 1, relative to sphere 2, in the (x,z)-plane. This is equivalent to changing the angle ψ or changing the direction of the z-axis by an amount $d\psi$, where $dx = (D_1+D_2)d\psi$. Hence the x-component of the force on sphere 1 is

$$\frac{1}{D_1 + D_2}\frac{\partial U}{\partial c} = \frac{1}{c(\coth\mu_1 + \coth\mu_2)}\frac{\partial U}{\partial \psi} \tag{43}$$

One might question the validity of our method of determining the electrostatic interaction energy of uncharged spheres when the spheres are far apart. We present the following arguments, considering only equal spheres at angle $\psi = 0$.

(i) The dipole moments of the uncharged spheres are unchanged when the origin of coordinates is moved. Thus if we have a set of charges e_i (such that $\Sigma_i e_i = 0$) at positions $\underset{\sim}{r}_i$, and shift the origin by $\underset{\sim}{h}$, then

$$\sum_i e_i(\underset{\sim}{r}_i+\underset{\sim}{h}) = \sum_i e_i\underset{\sim}{r}_i + \underset{\sim}{h}\sum_i e_i = \sum_i e_i\underset{\sim}{r}_i$$

Consider the two uncharged spheres at large separations. Then the dipole moment of sphere 1 with origin at centre of sphere 2 is equal to the dipole moment of sphere 1 with origin at its own centre. We show below that in the expansion

$$V_o(\mu,\eta,\phi) = -Ez + \frac{mz}{\epsilon_o r^3} + \ldots = -Ez + \frac{m\cos\theta}{\epsilon_o r^2} + \ldots \qquad (44)$$

the dipole moment m tends to the sum of the dipole moments of the two separate spheres, namely $2m_0 = 2\epsilon_o a^3 E$ for conducting spheres.

(ii) The equation (44) gives the first two terms of a multipole expansion for an electrically neutral distribution of charges in the presence of a uniform electric field. Choosing a convenient origin within the charge distribution, the expansion, equation (44), converges for values of r greater than the distance of the most distant charge from the origin. If the origin is taken midway between the centres of two equal spheres of radius a and separation R, then the multipole expansion converges for $r > R/2 + a$. As already mentioned, quadrupole and higher order multipole terms (proportional to r^{-3}, r^{-4}, etc.) do not contribute to the energy $-Em/2$ if the electric field is uniform.

(iii) For two equal spheres of radius a, $c = a\sinh\mu_0$, $R = 2a\cosh\mu_0$. For large R, $R \approx a\cdot\exp(\mu_0)$, $c \approx \frac{a}{2}\exp(\mu_0)$, so that $c \approx R/2$. Taking the origin midway between the two spheres, the multipole expansion is valid for $r > c+a$ at large separations.

(iv) Consider the relation

$$e^{\mu} = \frac{[1 + \frac{2zc}{r^2} + \frac{c^2}{r^2}]^{1/2}}{[1 - \frac{2zc}{r^2} + \frac{c^2}{r^2}]^{1/2}} \approx 1 + \frac{2zc}{r^2} + \frac{2zc^2}{r^4}(z-c) \qquad (45)$$

We showed that the first equation is equivalent to the equation of the sphere represented by the bispherical coordinate μ equal to a constant. The second equation gives an expansion in a binomial series. Putting $z = r\cos\theta$, the right-hand member in equation (45) can also be written as

$$1 + 2\frac{c}{r}\cos\theta + 2\frac{c^2}{r^2}\cos^2\theta - 2\frac{c^3}{r^3}\cos\theta$$

The expansion in a binomial series is valid provided

$$\left|\pm\frac{2zc}{r^2} + \frac{c^2}{r^2}\right| < 1 \ , \quad \text{i.e.} \ \left|\pm\frac{2c\cos\ell}{r} + \frac{c^2}{r^2}\right| < 1$$

This is permissible for sufficiently large r, say $r > 2c \approx R$, if R is large. Similar remarks may be made about the expansions of $\exp(-\mu)$, of $\exp[\pm(n+\frac{1}{2})\mu]$, of $\cos\eta$, and of $P_n(\cos\eta)$.

(v) For two equal spheres at angle $\psi = 0$, we showed that the dipole moment in equation (44) was

$$m = 4\sqrt{2}\epsilon_o c^2 \sum_{n=0}^{\infty} (n+\tfrac{1}{2})A_n = 8\epsilon_o E c^3 \sum_{n=0}^{\infty} (n+\tfrac{1}{2}) \frac{[2n+1-\alpha_0/(\beta_1+\beta_2)]}{e^{(2n+1)\mu_0} - 1} \tag{46}$$

where

$$\alpha_0 = \sum_{n=0}^{\infty} \frac{(2n+1)e^{-(2n+1)\mu_0}}{1 - e^{-(2n+1)\mu_0}}, \quad \beta_1+\beta_2 = \sum_{n=0}^{\infty} \frac{e^{-(2n+1)\mu_0}}{1 - e^{-(2n+1)\mu_0}} \tag{47}$$

(equations (5), (7) and (11)). We wish to show that when $R \to \infty$, $m \to 2m_0$, where $m_0 = \epsilon_o a^3 E$, the induced dipole moment of a single conducting sphere of radius a. At large R, we expand in powers of $\exp(-\mu_0) \approx \frac{a}{R}$, and after some algebra find that

$$\frac{\alpha_0}{\beta_1 + \beta_2} = 1 + 2e^{-2\mu_0} - 2e^{-3\mu_0} + 2e^{-4\mu_0} + 2e^{-5\mu_0} - 2e^{-6\mu_0} + \ldots \tag{48}$$

Hence the leading terms in A_0 and A_1 are

$$A_0 = -2\sqrt{2}Ece^{-3\mu_0}, \quad A_1 = 2\sqrt{2}Ece^{3\mu_0} \tag{49}$$

It follows immediately that after substituting $c = \frac{a}{2}\exp(-\mu_0)$ the leading term in m at large R is

$$m = 2\sqrt{2}\epsilon_o c^2 (A_0 + 3A_1) = 2Ea^3\epsilon_o = 2m_0 \tag{50}$$

We wish to find the leading term at large separations in the interaction energy of two uncharged perfectly conducting spheres of the same

radius a in the presence of an applied electric field. Then, since $A_n = -B_n$ when $\mu_1 = \mu_2 = \mu_0$, equation (28) yields

$$m_z = 4\sqrt{2}\epsilon_o c^2 \sum_{n=0}^{\infty} (n+\tfrac{1}{2})A_n = 2\sqrt{2}\epsilon_o c^2 (A_1 + 3A_2 + 5A_3 + \ldots) \tag{51}$$

where

$$A_n = \sqrt{2}Ec\cos\psi e^{-(2n+1)\mu_0}\left[2n + 1 - \frac{\alpha_0}{\beta_1+\beta_2}\right]\left[1 - e^{-(2n+1)\mu_0}\right]^{-1} \tag{52}$$

$\alpha_0/(\beta_1+\beta_2)$ is expressed as a power series in $\exp(-\mu_0)$ in equation (48). This yields

$$A_0 = -2\sqrt{2}Ec\cos\psi e^{-3\mu_0}\left[1 + e^{-2\mu_0} + 2e^{-3\mu_0} + \ldots\right]$$

$$A_1 = 2\sqrt{2}Ec\cos\psi e^{-3\mu_0}\left[1 - e^{-2\mu_0} + 2e^{-3\mu_0} + \ldots\right]$$

$$A_2 = 4\sqrt{2}Ec\cos\psi e^{-5\mu_0}(1 + \ldots)$$

and therefore

$$m_z = 16\epsilon_o Ec^3\cos\psi e^{-3\mu_0}\left[1 + 3e^{-2\mu_0} + 2e^{-3\mu_0} + \ldots\right] \tag{53}$$

From $c = a\sinh\mu_0$,

$$c^3 = (\tfrac{a}{2})^3 e^{3\mu_0}\left[1 - 3e^{-2\mu_0} + 3e^{-4\mu_0} + \ldots\right] \tag{54}$$

and so,

$$m_z = 2a^3\epsilon_o E\cos\psi\left[1 + 2e^{-3\mu_0} + \ldots\right] = 2a^3\epsilon_o E\cos\psi\left[1 + 2(\tfrac{a}{R})^3 + \ldots\right] \tag{55}$$

substituting $\exp(-\mu_o) \simeq a/R$. Thus

$$m_z = m_0\cos\psi + 4a^3\epsilon_o E(\tfrac{a}{R})^3\cos\psi + \ldots \tag{56}$$

The electrostatic energy associated with the dipole component m_z is

$$-\tfrac{1}{2}m_z E_z = -a^3\epsilon_o E^2\sin^2\psi - 2a^3(\tfrac{a}{R})^3\epsilon_o E^2\cos^2\psi + \ldots \tag{57}$$

This agrees with our multipole expansion for $\psi = 0$, the first term being the energy of polarization due to m_z of a single sphere in the electric field.

Since $C_n = D_n$ for equal radii, from equations (29) and (32)

$$m_x = 2\sqrt{2}c^2\epsilon_0 \sum_{n=1}^{\infty}(n^2+n)C_n = 2\sqrt{2}c^2\epsilon_0(C_1 + 3C_2 + \ldots) \tag{58}$$

where

$$C_n = 2\sqrt{2}Ec\sin\psi e^{-(2n+1)\mu_0}\left[1 + e^{-(2n+1)\mu_0}\right]^{-1} \tag{59}$$

Hence

$$C_1 = 2\sqrt{2}Ec\sin\psi e^{-3\mu_0}\left[1 - e^{-3\mu_0} + \ldots\right]$$

$$C_2 = 2\sqrt{2}Ec\sin\psi e^{-5\mu_0}(1 + \ldots)$$

and so

$$m_x = 16\epsilon_o Ec^3\sin\psi e^{-3\mu_0}\left[1 + 3e^{-2\mu_0} - e^{-3\mu_0} + \ldots\right] \tag{60}$$

Substituting equation (54) for c^3,

$$m_x = 2\epsilon_o Ea^3\sin\psi\left[1 - e^{-3\mu_0} + \ldots\right] = 2\epsilon_o Ea^3\sin\psi\left[1 - \left(\frac{a}{R}\right)^3 + \ldots\right] \tag{61}$$

The electrostatic energy associated with the dipole moment m_x is

$$-\frac{1}{2}m_x E_x = -a^3\epsilon_o E\sin^2\psi + a^3\left(\frac{a}{R}\right)^3\epsilon_o E^2\sin^2\psi + \ldots \tag{62}$$

the first term again being the energy of polarization due to m_x of a single sphere in the electric field. Thus, at large R the interaction energy is

$$a^3\left(\frac{a}{R}\right)^3\epsilon_o E^2(-2\cos^2\psi + \sin^2\psi) \tag{63}$$

This is the result obtained by Krasny-Ergen [8], although it is not clear whether he was dealing with uncharged spheres or with "earthed spheres". We have already discussed the difficulties in the case of two spheres at zero potentials at large separations. There would be no ambiguity if the spheres are uncharged.

APPENDIX C

Derivation of Equation (10)

Equation (1) can be written as

$$r^2 - 2zc\coth\mu + c^2\coth^2\mu = c^2(\coth^2\mu - 1) \tag{C1}$$

or

$$\coth\mu = \frac{r^2 + c^2}{2zc} \tag{C2}$$

Expanding in powers of 1/r,

$$e^{\mu} = \left[1 + \frac{2zc}{r^2} + \frac{c^2}{r^2}\right]^{1/2}\left[1 - \frac{2zc}{r^2} + \frac{c^2}{r^2}\right]^{-1/2}$$

$$= 1 + \frac{2zc}{r^2} + \frac{2c^2z}{r^4}(z-c) + \ldots \tag{C3}$$

Similarly,

$$e^{-\mu} = 1 - \frac{2zc}{r^2} + \frac{2c^2z}{r^4}(z+c) \tag{C4}$$

so that

$$\cosh\mu = 1 + \frac{2c^2z^2}{r^4} + \ldots \tag{C5}$$

Writing equation (C2) in the form

$$r^2 - 2c(r^2-z^2)^{1/2}\cot\eta = c^2 \quad \text{or} \quad \frac{\cos\eta}{(1-\cos^2\eta)^{1/2}} = \frac{r^2 - c^2}{2c(r^2-z^2)^{1/2}} \tag{C6}$$

we readily solve for $\cos\eta$ to obtain

$$\cos\eta = \frac{(1 - \frac{c^2}{r^2})}{\left[1 + \frac{c^2}{r^2} + 2\frac{cz}{r^2}\right]^{1/2}\left[1 + \frac{c^2}{r^2} - 2\frac{cz}{r^2}\right]^{1/2}} = 1 - 2\frac{c^2}{r^2} + 2\frac{c^2}{r^4}(c^2+z^2) + \ldots \tag{C7}$$

Hence

$$(\cosh\mu - \cos\eta)^{1/2} = \sqrt{2}\frac{c}{r}\left[1 - \frac{c^2}{2r^2} + \ldots\right] \tag{C8}$$

From equation (C3)

$$e^{(n+\frac{1}{2})\mu} = 1 + (2n+1)\frac{zc}{r^2} + (2n+1)\frac{zc^2}{r^4}\left[(n+\tfrac{1}{2})z - c\right] + \ldots \tag{C9}$$

and from equation (C4)

$$e^{-(n+\frac{1}{2})\mu} = 1 - (2n+1)\frac{zc}{r^2} + (2n+1)\frac{zc^2}{r^4}\left[(n+\tfrac{1}{2})z + c\right] + \ldots \tag{C10}$$

Hence

$$\sinh(n+\tfrac{1}{2})\mu = (2n+1)\frac{z\varsigma}{r^2} - (2n+1)\frac{z\varsigma^3}{r^4} + \ldots \tag{C11}$$

Using equation (C7), we expand

$$P_n(\cos\eta) = P_n(1) - 2\frac{\varsigma^2}{r^2}\left[\frac{dP_n(u)}{du}\right]_{u=1} + \ldots = 1 - 2\frac{\varsigma^2}{r^2}\left[\frac{dP_n(u)}{du}\right]_{u=1} + \ldots \tag{C12}$$

Proof of Equation (27)

We need to show that equation (27) holds, i.e.

$$\sum_{n=0}^{\infty} \frac{(2n+1)\left[e^{(2n+1)\mu_2} - e^{(2n+1)\mu_1}\right] + \lambda_3\left[e^{(2n+1)\mu_2} - 1\right] + \tau_3\left[e^{(2n+1)\mu_1} - 1\right]}{e^{(2n+1)(\mu_1+\mu_2)} - 1} = 0 \tag{C13}$$

where λ_3 and τ_3 are given in equation (21), in which

$$\beta_0 = \sum_{n=0}^{\infty} \frac{(2n+1)\left[e^{(2n+1)\mu_1} + 1\right]}{e^{(2n+1)(\mu_1+\mu_2)} - 1} \tag{C14}$$

$$\beta_1 = \sum_{n=0}^{\infty} \frac{e^{(2n+1)\mu_1}}{e^{(2n+1)(\mu_1+\mu_2)} - 1} \tag{C15}$$

$$\beta_2 = \sum_{n=0}^{\infty} \frac{1}{e^{(2n+1)(\mu_1+\mu_2)} - 1} \tag{C16}$$

and α_0, α_1 are obtained by replacing μ_1 by μ_2 in the numerators of the corresponding expressions for β_0 and β_1.

The series which occur here have the form

$$\sum_{n=0}^{\infty} \frac{(2n+1)^{\zeta}\, e^{(2n+1)\xi}}{e^{(2n+1)b} - 1} \tag{C17}$$

where $b = \mu_1 + \mu_2$, $\zeta = 0$ or 1, and $\xi = 0$, μ_1 or μ_2. Davis [13] has discussed these integrals in an Appendix, but seems to rely on a numerical calculation. It would be much better if an analytical approach were possible.

The proof of equation (27) is as follows. It is convenient to use the notation of Davis who defines the function $S_m(\xi)$ as in equation (31). Then our parameters α_i and β_i may be written as

$$\alpha_0 = S_1(\mu_2) + S_1(0) \ , \quad \alpha_1 = S_0(\mu_2) \tag{C18}$$

$$\beta_0 = S_1(\mu_1) + S_1(0) \ , \quad \beta_1 = S_0(0) \ , \quad \beta_2 = S_0(\mu_1) \tag{C19}$$

and

$$\lambda_3 = \frac{S_0(0)[S_1(\mu_1) + S_1(0)] - S_0(\mu_1)[S_1(\mu_2) + S_1(0)]}{S_0(\mu_2)S_0(\mu_1) - S_0^{\,2}(0)} \tag{C20}$$

$$\tau_3 = \frac{S_0(\mu_2)[S_1(\mu_1) + S_1(0)] - S_0(0)[S_1(\mu_2) + S_1(0)]}{S_0(\mu_2)S_0(\mu_1) - S_0^{\,2}(0)} \tag{C21}$$

Equation (27) or (C13) then reads

$$S_1(\mu_2) - S_1(\mu_1) + \lambda_3[S_0(\mu_2) - S_0(0)] + \tau_3[S_0(\mu_1) - S_0(0)] = 0 \tag{C22}$$

Substituting for λ_3 and τ_3, equation (31) is equivalent to

$$[S_1(\mu_2)-S_1(\mu_1)][S_0(\mu_2)S_0(\mu_1)-S_0^2(0)]$$

$$+ \Big[S_0(0)[S_1(\mu_1)+S_1(0)] - S_0(\mu_1)[S_1(\mu_2)-S_1(0)]\Big][S_0(\mu_2)-S_0(0)]$$

$$+ \Big[S_0(\mu_2)[S_1(\mu_1)+S_1(0)] - S_0(0)[S_1(\mu_2)+S_1(0)]\Big][S_0(\mu_1)+S_0(0)] = 0 \qquad \text{(C23)}$$

It is readily verified that equation (C23) can be written as the sum of ten pairs of terms, each of which is separately zero. This confirms the validity of our expansion, equation (18). It is striking that we need not sum the series $S_m(\xi)$.

V. INTERACTION OF TWO CONDUCTING SPHERICAL DROPLETS IN CONTACT IN A UNIFORM ELECTRIC FIELD

In the study of the electrostatic breaking of W/O emulsions under the previous contract [7], different methods of determining the force between two water droplets in a uniform electric field were considered. The most familiar approach is based on a multipole series expansion and gives as the leading term for uncharged spherical droplets a dipole-dipole attraction at large separations. For two equal perfectly conducting droplets with the line of centres parallel to the applied electric field, the force is

$$\frac{6\epsilon_o E^2 a^6}{R^4} \qquad (1)$$

where R is the distance between the centres of the droplets, ϵ_o the dielectric constant of the continuous (oil) medium and a is the droplet radius. This result is quoted by Stenzel and Eberz [24] and by Waterman [12]. However, the multipole expansion is not really suitable when the droplets are almost in contact, since the convergence of the series is too slow. Using a method of electrostatic imaging, Krasny-Ergen (8) has derived the interaction energy of two equal perfectly conducting spheres, whose line of centres is at an arbitrary angle to the electric field. The numerical calculations by Vlodavets et al. [9] of the results of Krasny-Ergen shows that equation (1) is an underestimate at small separations. Still another method, described by the writer in the above-mentioned report [7], is based on the use of bipolar co-ordinates, which should be more useful than the multipole series method at small separations between the droplets, although

its convergence at actual contact between the droplets is in question. Here we wish to examine the use of curvilinear so-called tangent-sphere coordinates for calculating the force between two perfectly conducting equal spherical droplets in contact with line of centres parallel to the uniform electric field [19-21].

Let ρ,z,λ denote cylindrical polar co-ordinates with origin at the point of contact O of the two droplets. Here z is the axis of symmetry which passes through O and the two droplet centres O_1 and O_2 with the electric field E parallel to the axis, ρ is the radial distance from the z axis in any direction normal to the axis, and λ is the rotation angle about the symmetry axis. We introduce the curvilinear orthogonal tangent-sphere co-ordinates ζ,ω,λ, defined by [17,18]

$$\zeta = \frac{z}{(\rho^2+z^2)}, \quad \omega = \frac{\rho}{(\rho^2+z^2)}, \quad \lambda = \lambda \tag{2}$$

or

$$z = \frac{\zeta}{(\zeta^2+\omega^2)}, \quad \rho = \frac{\omega}{(\zeta^2+\omega^2)}, \quad \lambda = \lambda \tag{3}$$

Both ζ and ω have dimensions of 1/distance. As shown in Figure 5, the ζ-coordinate surface is a sphere of radius $1/|2\zeta|$ touching the median plane z = 0 at O. The ω-coordinate surface is a torus formed by rotating about the z-axis a circle of radius $1/(2\omega)$ which is centred on the median plane z = 0 at a distance $\rho = 1/(2\omega)$ from the z-axis. The λ-coordinate surface is a plane passing through the z-axis. $\omega = 0$ is the z-axis, and $\zeta = 0$ is the median plane. At the origin $z = \rho = 0$, $\zeta \to \pm\infty$ and $\omega \to \infty$. Elements of length along the curvilinear ζ,ω,λ-axes are $h_\zeta d\zeta$, $h_\omega d\omega$, $h_\lambda d\lambda$ where

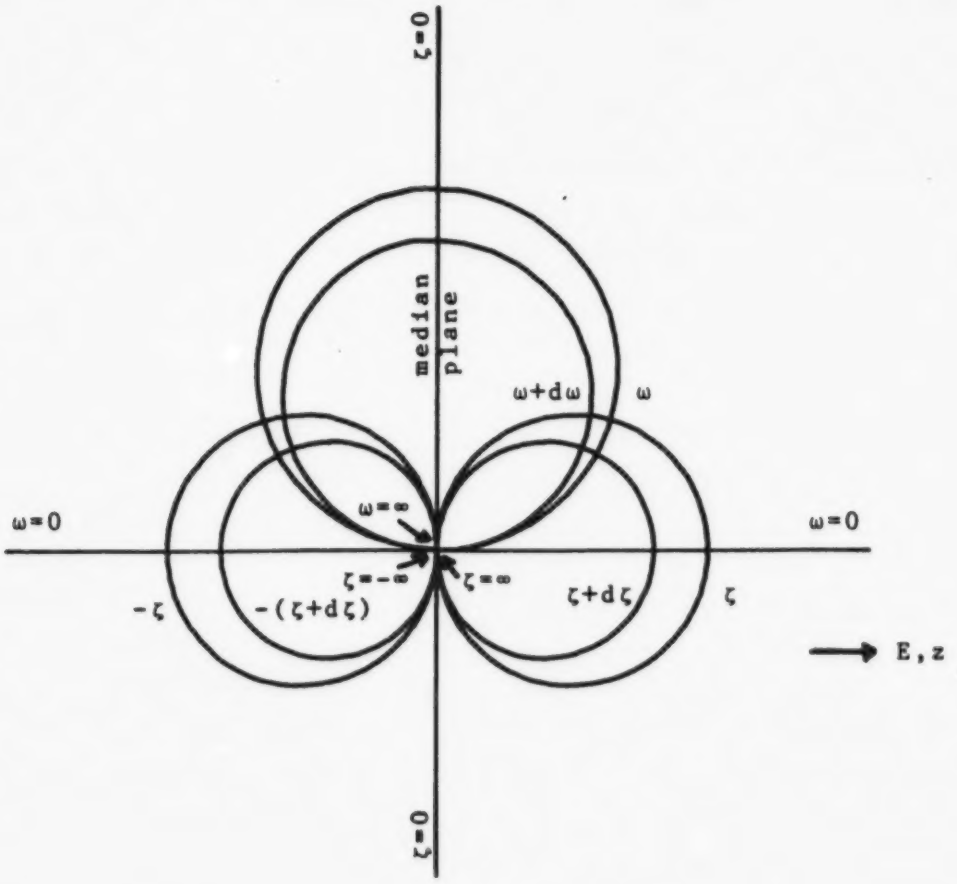

Figure 5. Tangent-sphere coordinates for two touching spheres.

$$h_\zeta = h_\omega = \frac{1}{\zeta^2+\omega^2}, \quad h_\lambda = \frac{\omega}{\zeta^2+\omega^2} \tag{4}$$

Since radial distance r from O is given by

$$r = (z^2 + \rho^2)^{1/2} = (\zeta^2 + \omega^2)^{-1/2} \tag{5}$$

both ζ and ω approach zero infinitely far from the two droplets. The droplet surfaces S_1 and S_2 are given by $\zeta = -\zeta_0$ and $\zeta = +\zeta_0$ respectively, where $\zeta_0 = 1/2a$, a being the droplet radius.

The electrostatic potential in the continuous oil medium satisfies Laplace's equation which reads, in tangent-sphere coordinates,

$$\nabla^2 V_o = \frac{(\zeta^2+\omega^2)^3}{\omega}\left[\frac{\partial}{\partial\omega}\left(\frac{\omega}{\zeta^2+\omega^2}\frac{\partial V_o}{\partial\omega}\right) + \omega\frac{\partial}{\partial\zeta}\left(\frac{1}{\zeta^2+\omega^2}\frac{\partial V_o}{\partial\omega}\right)\right] = 0 \tag{6}$$

At large distances, V_o tends to the uniform electric field

$$V_o = -Ez = \frac{-E\zeta}{(\zeta^2+\omega^2)} \tag{7}$$

Substitution into equation (6) of

$$V_o = \rho^{-1/2}U(\zeta)W(\omega) = \left[\frac{\zeta^2+\omega^2}{\omega}\right]^{1/2}U(\zeta)W(\omega) \tag{8}$$

yields

$$\frac{d^2U(\zeta)}{d\zeta^2} - g^2U(\zeta) = 0 \tag{9}$$

and

$$\frac{d^2W(\omega)}{d\omega^2} + (\frac{1}{4\omega^2} + g^2)W(\omega) = 0 \qquad (10)$$

where g is an arbitrary constant having the dimensions of distance. Noting that $W(\omega) = \omega^{1/2}J_0(g\omega)$ where $J_0(g\omega)$ denotes the Bessel function of the first kind of order zero, the solution of equation (6) which tends to the limiting form in equation (7) at large distances may be expressed as

$$V_o(\zeta,\omega) = \frac{-E\zeta}{(\zeta^2+\omega^2)} + (\zeta^2+\omega^2)^{1/2}\int_0^\infty \left[A(g)e^{\zeta g} + B(g)e^{-\zeta g}\right]gJ_0(g\omega)dg \qquad (11)$$

where A(g) and B(g) are functions to be determined by the appropriate boundary conditions. The integral part on the right-hand side of equation (11) should remain finite at infinity. We first consider uncharged droplets, in which case the potential $V_o(\zeta,\omega)$ on the two equal droplets at $\zeta = \zeta_0$ (S_2) and $\zeta = -\zeta_0$ (S_1) can be put equal to zero, so that we have as boundary conditions

$$V_o(\zeta_0,\omega) = V_o(-\zeta_0,\omega) = 0 \qquad (12)$$

Substituting firstly $\zeta = \zeta_0$ and secondly $\zeta = -\zeta_0$ into equation (11) and then adding the two resulting equations, it follows on using equation (12) that

$$\int_0^\infty [A(g) + B(g)]\cosh(g\zeta_0)gJ_0(g\omega)dg = 0 \qquad (13)$$

Performing a Fourier-Bessel inversion* yields

$$A(g) + B(g) = 0 \tag{14}$$

and therefore we can write the solution of equation (6) in the form

$$V_o(\zeta,\omega) = \frac{-E\zeta}{(\zeta^2+\omega^2)} + (\zeta^2+\omega^2)^{1/2}\int_0^\infty A(g)\sinh(g\zeta)gJ_0(g\omega)dg \tag{15}$$

where we have absorbed a factor of 2 into A(g). Fourier-Bessel inversion of the condition $V_o(\zeta_0,\omega) = 0$ yields

$$A(g)\sinh(g\zeta_0) = E\zeta_0\int_0^\infty \frac{\omega J_0(g\omega)}{(\zeta_0{}^2+\omega^2)^{3/2}}d\omega = Ee^{-\zeta_0 g} \tag{16}$$

Hence the required potential is

$$V_o(\zeta,\omega) = \frac{-E\zeta}{(\zeta^2+\omega^2)} + E(\zeta^2+\omega^2)^{1/2}\int_0^\infty e^{-\zeta_o g}\frac{\sinh(g\zeta)}{\sinh(g\zeta_o)}gJ_o(g\omega)dg \tag{17}$$

The force on the droplet surface S_2 $(\zeta=\zeta_0)$ due to the droplet S_1 $(\zeta=-\zeta_0)$ and the applied electric field acting along the line of centres of the

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* This inversion is

$$G(\omega) = \int_0^\infty F(g)gJ_0(g\omega)dg\ , \quad F(g) = \int_0^\infty G(\omega)\omega J_0(g\omega)d\omega$$

two droplets (i.e. in the direction of the z-axis), may be expressed in terms of the Maxwell electrostatic stress tensor in the form [16]

$$F = \frac{\epsilon}{4\pi} \int_{S_2} \left[X_n (\underset{\sim}{X} \cdot \underset{\sim}{k}) - \frac{1}{2} X^2 (\underset{\sim}{n} \cdot \underset{\sim}{k}) \right] dS \tag{18}$$

where $F < 0$ for an attractive force. Here $\underset{\sim}{X} = -\text{grad } V_o$ is the electric field vector having the magnitude X and the outward normal component X_n to S_2, $\underset{\sim}{n}$ is the unit outward normal vector to S_2, and $\underset{\sim}{k}$ is the unit vector along the z-axis. Since S_2 $(\zeta = \zeta_0)$ is an equipotential surface and ζ is decreasing outwards from S_2 (Figure 5)

$$X = |X_n| \ , \quad X_n = \left(\frac{1}{h_\zeta} \frac{\partial V_o}{\partial \zeta} \right)_{\zeta = \zeta_0} \tag{19}$$

Also

$$\underset{\sim}{n} \cdot \underset{\sim}{k} = -\frac{1}{h_\zeta} \frac{\partial z}{\partial \zeta} \ , \quad \underset{\sim}{X} \cdot \underset{\sim}{k} = X_n (\underset{\sim}{n} \cdot \underset{\sim}{k}) \tag{20}$$

and an element of area on S_2 is $dS = h_\lambda h_\omega d\lambda d\omega$. Hence, integrating with respect to λ from 0 to 2π and with respect to ω from 0 to ∞, equation (18) becomes

$$F = -\frac{\epsilon}{4} \int_0^\infty \left[\frac{h_\omega h_\lambda}{h_\zeta^3} \left(\frac{\partial V_o}{\partial \zeta} \right)^2 \frac{\partial z}{\partial \zeta} \right]_{\zeta = \zeta_0} d\omega \tag{21}$$

By equations (3) and (4)

$$\left[\frac{h_\omega h_\lambda}{h_\zeta^3}\frac{\partial z}{\partial \zeta}\right]_{\zeta=\zeta_0} = -\omega\frac{(\zeta_0^2-\omega^2)}{(\zeta_0^2+\omega^2)} \tag{22}$$

Also, using the integral (Ref. [25], p. 712)

$$\int_0^\infty g^n e^{-rg} J_0(gx)dg = \begin{cases} \dfrac{1}{(r^2+x^2)^{1/2}}, & n=0 \\ \\ (-1)^n \dfrac{d^n}{dr^n}\left[(r^2+x^2)^{-1/2}\right], & n=1,2,\ldots \end{cases} \tag{23}$$

the equation (17) yields

$$\left(\frac{\partial V_o}{\partial \zeta}\right)_{\zeta=\zeta_0} = E\left[\frac{2\zeta_0^2-\omega^2}{(\zeta_0^2+\omega^2)^2} + \frac{(\zeta_0^2+\omega^2)^{1/2}}{\zeta_0^3} I\left(\frac{\omega}{\zeta_0}\right)\right] \tag{24}$$

where

$$I(x) = \int_0^\infty g^2 \coth(g) e^{-g} J_0(gx)dg \tag{25}$$

We expand

$$\coth(g) = \frac{1+e^{-2g}}{1-e^{-2g}} = 1 + 2\sum_{n=1}^{\infty} e^{-2ng} \tag{26}$$

which is valid for $g > 0$. Consider therefore

$$\lim_{\delta\to 0}\int_{\delta}^{\infty} g^2\coth(g)e^{-g}J_0(gx)dg$$

$$-\lim_{\delta\to 0}\left[\int_{\delta}^{\infty} g^2e^{-g}J_0(gx)dg + 2\sum_{n=1}^{\infty}\int_{\delta}^{\infty} g^2J_0(gx)e^{-(2n+1)g}dg\right] \tag{27}$$

Assume that we can interchange the orders of summation and taking the limit $\delta \to 0$ (see Appendix D). Then, using equation (23), we obtain

$$I(x) = \frac{2 - x^2}{(1+x^2)^{5/2}} + 2\sum_{n=1}^{\infty}\frac{2(2n+1)^2 - x^2}{\left[(2n+1)^2+x^2\right]^{5/2}} \tag{28}$$

The series converges since the terms behave as $1/n^3$ for large n. Substituting equations (22) and (24) into equation (21) we obtain

$$F = \frac{\epsilon E^2}{4\zeta_0^2}\left[\frac{1}{4} + 2\int_0^{\infty}\frac{x(1-x^2)(2-x^2)}{(1+x^2)^{5/2}}I(x)dx + \int_0^{\infty} x(1-x^2)I^2(x)dx\right] \tag{29}$$

We indicate how the above theory can be extended to two equal touching spheres, each carrying a charge q. In place of equation (12) we have

$$V_o(\zeta_0,\omega) = V_o(-\zeta_0,\omega) = \phi_o \tag{30}$$

where ϕ_o is the common potential of the two droplets. Again substituting $\zeta = \zeta_0$ and then $\zeta = -\zeta_0$ into equation (11) and adding the two resulting equations, use of equation (30) now yields in place of equation (13)

$$\int_0^\infty [A(g) + B(g)]\cosh(\zeta_0 g) g J_0(g\omega) dg = \frac{\phi_o}{(\zeta_0^2+\omega^2)^{1/2}} \tag{31}$$

By Fourier-Bessel inversion,

$$[A(g) + B(g)]\cosh(\zeta_0 g) = \phi_o \int_0^\infty \frac{\omega J_0(g\omega)}{(\zeta_0^2+\omega^2)^{1/2}} d\omega = \phi_o \frac{e^{-\zeta_0 g}}{g} \tag{32}$$

Expressing B(g) in terms of A(g), we can write equation (11) as

$$V_o(\zeta,\omega) = \frac{-E\zeta}{(\zeta^2+\omega^2)} + (\zeta^2+\omega^2)^{1/2} \int_0^\infty A(g)\sinh(g\zeta) g J_0(g\omega) dg$$

$$+ \phi_o(\zeta^2+\omega^2)^{1/2} \int_0^\infty \frac{e^{-g(\zeta+\zeta_0)}}{\cosh(\zeta_0 g)} J_0(g\omega) dg \tag{33}$$

which, of course, reduces to equation (15) on putting $\phi_0 = 0$. Similarly to the treatment of equation (25), we expand

$$\frac{e^{-g(\zeta+\zeta_0)}}{\cosh(\zeta_0 g)} = \frac{2e^{-g(\zeta+2\zeta_0)}}{1 + e^{-2\zeta_0 g}} = 2e^{-g(\zeta+2\zeta_0)} \sum_{n=0}^{\infty} (-1)^n e^{-2n\zeta_0 g} \tag{34}$$

and use equation (23) to express the last term on the right-hand side of (33) as

$$2\phi_o(\zeta^2+\omega^2)^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n}{\left[(\zeta+2(n+1)\zeta_0)^2 + \omega^2\right]^{1/2}} \tag{35}$$

Fourier-Bessel inversion of the condition $V_o(\zeta_0,\omega) = \phi_o$ leads to

$$A(g)\sinh(g\zeta_0) = \int_0^\infty \omega J_0(g\omega)d\omega \left[\frac{\phi_o}{(\zeta_0^2+\omega^2)^{1/2}} + \frac{E\zeta_o}{(\zeta_0^2+\omega^2)^{3/2}}\right.$$

$$\left. - 2\phi_o \sum_{n=0}^{\infty} \frac{(-1)^n}{\left[(2(n+3)\zeta_0)^2 + \omega^2\right]^{1/2}}\right] \qquad (36)$$

so that A(g) is determined, and hence $V_o(\zeta,\omega)$. The force on S_2 is given by equation (21), which we shall not develop here. The relation between the potential and the charge on a droplet is obtained from Gauss's theorem, which in tangent-sphere co-ordinates reads

$$\int_0^\infty \left[\frac{h_\omega h_\lambda}{h_\zeta} \frac{\partial V_o(\zeta,\omega)}{\partial\zeta}\right]_{\zeta=\zeta_0} d\omega = \frac{2q}{\epsilon_o} \qquad (37)$$

It should be possible to use tangent-sphere coordinates to determine the interaction energy of two conducting spheres in contact when the electric field is applied at right angles to the line of centres of the spheres, and then to consider the line of centres at arbitrary angle to the electric field. Comparison can then be made with the work of Krasny-Ergen [8]. Also the two spheres can have different radii (see Figure 5) although they must be at the same potential if in contact. Numerical evaluation of the integrals or of their expressions in terms of infinite series should be quite feasible (see [21]). The advantage of the tangent-sphere co-ordinates is that exact expressions are obtained for two conducting spheres in contact.

APPENDIX D

The series in equation (26) diverges at the lower limit of integration $g = 0$ in equation (27) and therefore it is desirable to confirm the validity of interchanging orders of series summation and taking the limit $\delta \to 0$ in equation (27). We need to consider the limit of the second term on the right-hand side of equation (27), which can be written as

$$\lim_{\delta \to 0} \sum_{n=1}^{\infty} \left[\int_0^{\infty} g^2 J_0(gx) e^{-(2n+1)g} dg - \int_0^{\delta} g^2 J_0(gx) e^{-(2n+1)g} dg \right] \tag{D1}$$

The first integral does not depend on δ and is given by equation (23). Hence we need to show that

$$\lim_{\delta \to 0} \sum_{n=1}^{\infty} \int_0^{\delta} g^2 J_0(gx) e^{-(2n+1)g} dg = 0 \tag{D2}$$

$J_0(gx)$ is a decaying oscillatory function of gx, such that $|J_0(gx)| \leq 1$ at large gx and $J_0(gx) \sim (2/\pi gx)^{1/2} \cos(gx - \pi/4)$ at large gx. $g^2 \exp[-(2n+1)g]$ is a monotonically decreasing function of g for values of $g > 2/[(2n+1)]$ where it is maximum. The first zero of $J_0(gx)$ is at $g = 2.4/x$. Hence for $2/[(2n+1)] < 2.4/x$, or $2n+1 > x/(1.2)$, the factor $g^2 \exp[-(2n+1)]$ will definitely reduce the oscillations in $J_0(gx)$. We can omit a finite number, say N, of terms in equation (D2), since the limit of their sum as $\delta \to 0$ will be zero. So we need only show that

$$\lim_{\delta \to 0} \sum_{n=N}^{\infty} \int_0^{\delta} g^2 J_0(gx) e^{-(2n+1)g} dg = 0 \tag{D3}$$

where N is a suitable finite integer. We can safely assume that

$$\int_0^{\delta} g^2 J_0(gx) e^{-(2n+1)g} dg > 0 \tag{D4}$$

since the spacing between successive zeros of $J_0(gx)$ are approximately equal. The integrand is positive in the first interval of $J_0(gx)$, $0 \leq g \leq 2.4/x$, and for large enough n this interval will determine the sign of the integral in equation (D4) (see Figure 6). Thus we write

$$0 < \int_0^{\delta} g^2 J_0(gx) e^{-(2n+1)g} dg < \int_0^{\delta} g^2 |J_0(gx)| e^{-(2n+1)g} dg$$

$$< \int_0^{\delta} g^2 e^{-(2n+1)g} dg \tag{D5}$$

where

$$\int_0^{\delta} g^2 e^{-(2n+1)g} dg = -\left[\frac{\delta^2}{(2n+1)} + \frac{2\delta}{(2n+1)^2}\right] e^{-(2n+1)\delta}$$

$$+ \frac{2}{(2n+1)^3}\left[1 - e^{-(2n+1)\delta}\right] \tag{D6}$$

We proceed to sum each term on the right-hand side of equation (D6).

$$\text{(i)} \quad \delta^2 \sum_{n=N}^{\infty} \frac{e^{-(2n+1)\delta}}{2n+1} < \delta^2 \sum_{n=0}^{\infty} \frac{e^{-(2n+1)\delta}}{2n+1} < \delta^2 \sum_{n=0}^{\infty} e^{-(2n+1)\delta}$$

$$- \delta^2 \left[\frac{e^{-\delta}}{1 - e^{-2\delta}} \right] \to 0 \quad \text{as} \quad \delta \to 0$$

since $1 - \exp(-2\delta) \to 2\delta$ as $\delta \to 0$.

(ii) $$2\delta \sum_{n=N}^{\infty} \frac{e^{-(2n+1)\delta}}{(2n+1)^2} \to 0 \text{ as } \delta \to 0$$

noting that the series converges for $\delta \to 0$.

(iii) Using the inequality $1 - e^{-x} < x$ for $x > 0$

$$2 \sum_{n=N}^{\infty} \frac{1 - e^{-(2n+1)\delta}}{(2n+1)^3} < 2 \sum_{n=N}^{\infty} \frac{(2n+1)\delta}{(2n+1)^3} \to 0 \text{ as } \delta \to 0$$

observing again that the series converges. The required condition in equation (D3) has been proved.

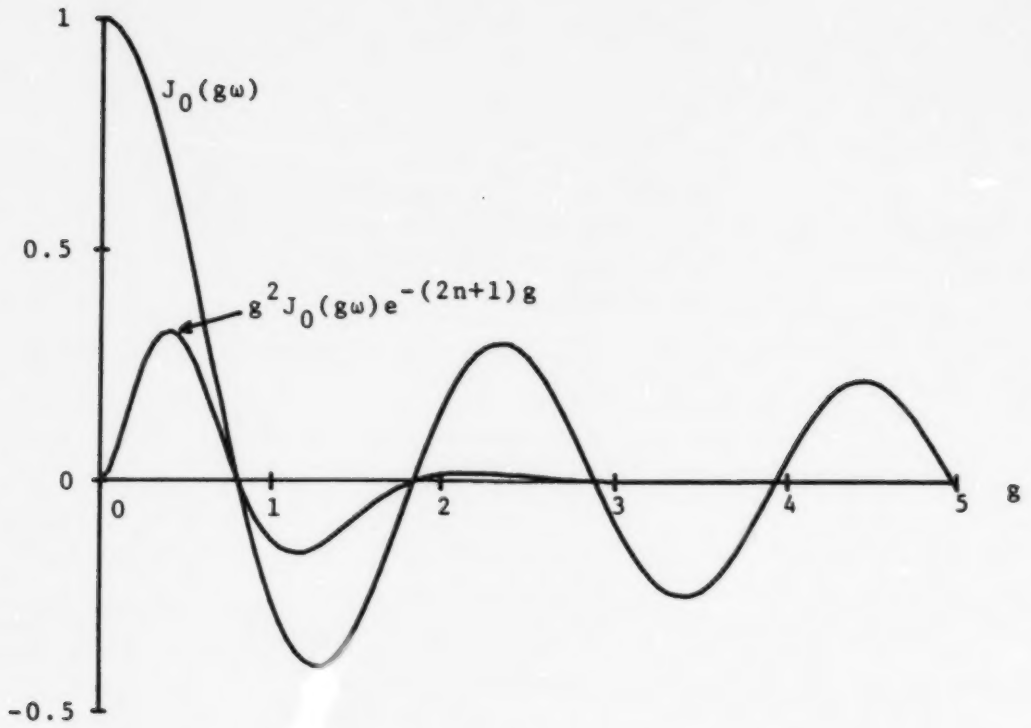

Figure 6. Plot of the functions $J_0(g\omega)$ and $g^2 J_0(g\omega)e^{-(2n+1)g}$ for $\omega = 3$ and $n = 1$. The scale for the latter function has been increased by a factor of 10.

VI. DISTORTION OF WATER DROPLET IN ELECTRIC FIELD

In an electric field a water droplet suspended in an oil medium and carrying no net charge elongates in the direction of the applied field. To a first approximation, the droplet assumes the shape of a prolate spheroid. We shall consider two models, a dielectric spheroid and a perfectly conducting spheroid. Using the notation of Abramowitz and Stegun [26], we introduce prolate spheroidal co-ordinates ξ,η,ϕ, which are related to Cartesian coordinates x,y,z by

$$x = f\left[(\xi^2-1)(1-\eta^2)\right]^{1/2}\cos\phi, \quad y = f\left[(\xi^2-1)(1-\eta^2)\right]^{1/2}\sin\phi, \quad z = f\xi\eta \qquad (1)$$

where the applied electric field is in the direction of the z-axis, ξ = constant represents an ellipsoid of revolution about this axis, and η = constant an orthogonal hyperboloid of revolution about the same axis. The two families of surfaces ξ = constant and η = constant all have the same foci on the z-axis. The distance between the two foci is 2f, the two foci having Cartesian coordinates x = 0, y = 0, z = $\pm$f. For a typical ellipsoid, the semi-major axis a, the semi-minor axis b, and the eccentricity e are given by

$$a = f\xi, \quad b = f[\xi^2-1]^{1/2}, \quad e = f/a = 1/\xi \qquad (2)$$

ξ varies from 1 to ∞, η from -1 to +1, and ϕ from 0 to 2π (see Figure 7). For the prolate ellipsoid of revolution representing the water droplet with the applied field E along the axis of symmetry (z), the potential V is

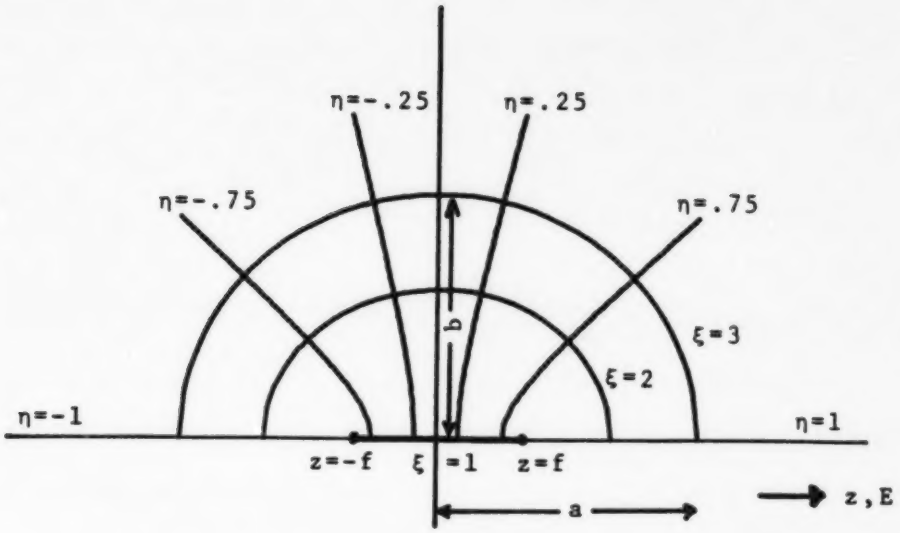

Figure 7. Prolate spheroidal coordinates.

independent of angle ϕ and the Laplace equation for V becomes

$$\nabla^2 V = \frac{\partial}{\partial \xi}\left[(\xi^2-1)\frac{\partial V}{\partial \xi}\right] + \frac{\partial}{\partial \eta}\left[(1-\eta^2)\frac{\partial V}{\partial \eta}\right] = 0 \tag{3}$$

Let V_i and V_o be the electrostatic potentials inside and outside the ellipsoid which is defined by a given value of ξ, say $\xi = \xi_o$. Separation of variables ξ and η yields the general solution of equation (3) to be

$$V = \sum_{n=0}^{\infty} [A_n P_n(\xi) + B_n Q_n(\xi)][C_n P_n(\eta) + D_n Q_n(\eta)] \tag{4}$$

At large distances from the ellipsoid V_o, tends to an external potential describing the electric field parallel to the z-axis, namely

$$V_o \rightarrow -Ez = -Ef\xi\eta \tag{5}$$

In equation (4) P_n and Q_n are Legendre functions of integral order n of the first and second kind. The limiting form (5) requires that the Legendre polynomial P_n be confined to n = 0 and n = 1. Also $Q_n(\eta)$ has a logarithmic singularity at $\eta = 1$ so that we must choose $D_n = 0$, and since the potential is defined to an additive constant, we can put $A_0 = 0$. We end up in the outer (oil) region with the expression

$$V_o = B_0 Q_o(\xi) + [A_1 \xi + B_1 Q_1(\xi)]\eta \tag{6}$$

since $P_0(\eta) = 1$, $P_1(\xi) = \xi$, $P_1(\eta) = \eta$, and the factors C_0 and C_1 are conveniently absorbed into the other constants, so that V_0 involves three constants to be determined from the boundary conditions.

Electric Field Inside an Uncharged Dielectric Ellipsoid

Inside the ellipsoidal droplet (where $1 \leq \xi \leq \xi_o$) both ξ and η can take the value 1 so that $Q_n(\eta)$ and $Q_n(\xi)$ are both absent in equation (4). Again n only assumes values 0 and 1. Using lower-case letters for constants, the potential V_i is

$$V_i = a_0 + a_1\xi\eta \tag{7}$$

We shall first apply the boundary condition (5). We have that

$$Q_0(\xi) = \frac{1}{2}\ln\left[\frac{\xi+1}{\xi-1}\right] = \coth^{-1}\xi, \quad Q_1(\xi) = \frac{1}{2}\xi\ln\left[\frac{\xi+1}{\xi-1}\right] - 1 = \xi\coth^{-1}\xi - 1 \tag{8}$$

Expanding in powers of $1/\xi$ at large ξ, where equation (5) applies

$$Q_0(\xi) \approx \frac{1}{\xi} + \frac{1}{3\xi^3} + \ldots, \quad Q_1(\xi) = \frac{1}{3\xi^2} + \frac{1}{5\xi^4} + \ldots \tag{9}$$

On equating equation (6) to equation (5) at large ξ, it follows that

$$A_1 = -Ef \tag{10}$$

Let ϵ_i and ϵ_o be the dielectric constants of the water and oil respectively. Then, assuming no surface charge on the emulsion droplet, the two boundary conditions at the oil/water interface $\xi = \xi_o$ are

$$V_i = V_o, \quad \frac{\epsilon_o}{h_\xi}\frac{\partial V_o}{\partial \xi} = \frac{\epsilon_i}{h_\xi}\frac{\partial V_i}{\partial \xi} \tag{11}$$

where the metric coefficients in prolate spheroidal co-ordinates are

$$h_\xi = f\left[\frac{\xi^2-\eta^2}{\xi^2-1}\right]^{1/2}, \quad h_\eta = f\left[\frac{\xi^2-\eta^2}{1-\eta^2}\right]^{1/2}, \quad h_\phi = f\left[(\xi^2-1)(1-\eta^2)\right]^{1/2} \tag{12}$$

Substituting equations (6) and (7) into the first condition in equation (11), we obtain

$$B_0Q_o(\xi_o) + [-Ef\xi_o + B_1Q_1(\xi_o)]\eta = a_0 + a_1\xi_o\eta \tag{13}$$

from which it follows that

$$B_0Q_0(\xi_o) = a_0 \tag{14}$$

and

$$-Ef\xi_o + B_1Q_1(\xi_o) = a_1\xi_o \tag{15}$$

Similarly the second condition in equation (11) yields

$$\epsilon_o\left[B_0\frac{dQ_0(\xi_o)}{d\xi_o} + \left[-\,Ef + B_1\frac{dQ_1(\xi_o)}{d\xi_o}\right]\eta\right] - \epsilon_i a_1\eta \tag{16}$$

from which, on combining with equation (14), we derive

$$B_0 - a_0 - 0 \tag{17}$$

and

$$\epsilon_o\left[-\,Ef + B_1\frac{dQ_1(\xi_o)}{d\xi_o}\right] - \epsilon_i a_i \tag{18}$$

Solving equations (15) and (18) for B_1 and a_1,

$$B_1 - \frac{Ef(\epsilon_i-\epsilon_o)}{\frac{\epsilon_i}{\xi_o}Q_1(\xi_o) - \epsilon_o\frac{dQ_1(\xi_o)}{d\xi_o}} - \frac{Ef(\epsilon_i-\epsilon_o)}{\frac{\epsilon_o\xi_o}{(\xi_o^2-1)} + \left[\frac{\epsilon_i}{\xi_o} - \epsilon_o\right]\coth^{-1}\xi_o} \tag{19}$$

and

$$a_1 - \frac{Ef\epsilon_o\left[\xi_o\frac{dQ_1(\xi_o)}{d\xi_o} - Q_1(\xi_o)\right]}{\epsilon_i Q_1(\xi_o) - \epsilon_o\xi_o\frac{dQ_1(\xi_o)}{d\xi_o}} - \frac{-Ef\epsilon_o}{\xi_o(\epsilon_i-\epsilon_o)\left[(\xi_o^2-1)\coth^{-1}\xi_o - \xi_o\right] + \epsilon_i} \tag{20}$$

The electric field intensity inside the ellipsoid is given by

$$E_i^2 - \left[\frac{1}{h_\xi}\frac{\partial V_i}{\partial\xi}\right]^2 + \left[\frac{1}{h_\eta}\frac{\partial V_i}{\partial\eta}\right]^2 - a_1^2\left[\frac{\eta^2}{h_\xi^2} + \frac{\xi^2}{h_\eta^2}\right]^2 - \frac{a_1^2}{f^2} \tag{21}$$

i.e. the field intensity, $E_i = -a_1 f$, is uniform. The results stated in equations (20) and (21) are identical with those quoted in Smythe [27].

Potential Inside a Perfectly Conducting Charged Ellipsoid

Let us now consider a perfectly conducting ellipsoid as the model of an emulsion droplet under the influence of an electric field. The spheroidal surface carries a total charge q at potential Φ. Thus the potential V_o in the oil medium defined in equation (6) satisfies the boundary condition

$$V_o = \Phi \quad \text{at} \quad \xi = \xi_o \tag{22}$$

Since Φ is a constant, the coefficient of η in equation (6) vanishes at $\xi = \xi_o$. A_1 is again given by equation (10) and hence

$$B_1 = \frac{-A_1 \xi_o}{Q_1(\xi_o)} = \frac{Ef\xi_o}{Q_1(\xi_o)} \tag{23}$$

When the applied electric field is in the direction of the major axis of the ellipsoid, the surface charge density on the ellipsoid will be symmetrical about this axis (z-axis) and may be denoted by $\sigma(\eta)$. On applying Gauss's theorem

$$\epsilon_o \frac{\partial V_o}{\partial n} = \frac{\epsilon_o}{h_\xi} \frac{\partial V_o}{\partial \xi} = -4\pi\sigma(\eta) \quad \text{at } \xi = \xi_o \tag{24}$$

where $\partial/\partial n$ denotes differentiation along the outward normal to the ellipsoid $\xi = \xi_o$. An element of area on the ellipsoid is given by $dS = h_\phi h_\eta d\phi d\eta$. Substituting equation (6) into equation (24)

$$\frac{\epsilon_o}{f}\left[\frac{\xi_o^2-1}{\xi_o^2-\eta^2}\right]^{1/2}\left[B_0\frac{dQ_0(\xi_o)}{d\xi_o}+\left[A_1+B_1\frac{dQ_1(\xi_o)}{d\xi_o}\right]\eta\right]=-4\pi\sigma(\eta) \tag{25}$$

Integration of the charge density over the total area of the ellipsoid yields q, namely

$$q=\int_0^{2\pi}d\phi\int_{-1}^{1}h_\phi h_\eta\sigma(\eta)d\eta=\frac{-\epsilon_o}{4\pi}\int_0^{2\pi}d\phi\int_{-1}^{1}\frac{h_\phi h_\eta}{h_\xi}\frac{\partial V_o}{\partial\xi}\,d\eta$$

$$=\frac{-\epsilon_o f}{2}(\xi_o^2-1)\int_{-1}^{1}\left[B_0\frac{dQ_0(\xi_o)}{d\xi_o}+\left[A_1\xi_o+B_1\frac{dQ_1(\xi_o)}{d\xi_o}\right]\eta\right]d\eta \tag{26}$$

$$=-\epsilon_o f(\xi_o^2-1)B_0\frac{dQ_0(\xi_o)}{d\xi_o}=\epsilon_o fB_0$$

so that B_0 is determined. We note that $B_0 = 0$ if $q = 0$.

If the dielectric model for the ellipsoidal droplet also includes a surface charge $\sigma(\eta)$, then the second boundary condition in equation (11) is replaced by

$$\frac{\epsilon_o}{h_\xi}\frac{\partial V_o}{\partial\xi}-\frac{\epsilon_i}{h_\xi}\frac{\partial V_i}{\partial\xi}=-4\pi\sigma(\eta)\quad\text{at}\quad\xi=\xi_o \tag{27}$$

The solutions for the constants in the potentials V_o and V_i become more complicated and the field intensity E_i inside the ellipsoid is no longer uniform. For the water-in-oil emulsions with ϵ_i/ϵ_o of order 25, the approximation of the simpler perfectly conducting ellipsoid carrying a surface charge should be adequate at this stage, particularly if the aqueous interior is an electrolyte.

Surface Free Energy of an Ellipsoid

Given the volume of the droplet, we wish to find the shape of the ellipsoid (i.e. the eccentricity e) at stable equilibrium of the droplet in the presence of the applied electric field. This requires determining the minimum in the free energy associated with the surface tension of the oil/water interface and the electrostatic field E. If γ is the surface tension of the oil/water interface, then the associated energy that is required equals γ times the area of the ellipsoid, namely

$$F_s = \gamma\int_0^{2\pi} d\phi \int_{-1}^{1} h_\phi h_\eta d\eta = \gamma f^2(\xi_o^2-1)^{1/2}2\pi\int_{-1}^{1}(\xi_o^2-\eta^2)^{1/2}d\eta$$

$$= 2\pi\gamma f^2(\xi_o^2-1)^{1/2}\left[(\xi_o^2-1)^{1/2} + \xi_o^2\sin^{-1}(\frac{1}{\xi_o})\right] \tag{28}$$

Introducing the eccentricity $e = 1/\xi_o$ and noting that the major and minor axes are related by $a(1-e^2)^{1/2} = b$, we can write the result in equation (28) as

$$F_s = 2\pi\gamma ba(1-e^2)^{1/2}\left[1 + \frac{\sin^{-1}e}{e(1-e^2)^{1/2}}\right] \tag{29}$$

This result is also given by Panchenkov and Tsabek [28,29] who obtain F_s rather differently as described in Appendix E. Since the volume of an ellipsoid is

$$\Omega = \frac{4}{3}\pi ab^2 = \frac{4}{3}\pi\frac{(1-e^2)}{e^3}f^3 \tag{30}$$

equation (29) can be expressed as

$$F_s = 2\pi\gamma\left[\frac{3\Omega}{4\pi}\right]^{2/3}(1-e^2)^{1/3}\left[1 + \frac{\sin^{-1}e}{e(1-e^2)^{1/2}}\right] \tag{31}$$

Potential Outside a Perfectly Conducting Charged Ellipsoid

For a perfectly conducting ellipsoid, equation (6) for the potential in the oil medium is

$$V_o = \frac{q}{\epsilon_o f}Q_0(\xi) + Ef\eta\left[-\xi + \frac{\xi_o Q_1(\xi)}{Q_1(\xi_o)}\right] \tag{32}$$

We wish to examine the limiting form of V_o at large distances from the ellipsoid. The origin of Cartesian co-ordinates x,y,z in equation (1) is midway between the two foci. If R is the distance to a general point x,y,z from the origin and r_1 and r_2 are the corresponding distances from the two foci, then

$$\begin{aligned} r_1^2 &= x^2 + y^2 + (z-f)^2 = R^2 - 2zf + f^2 \\ r_2^2 &= x^2 + y^2 + (z+f)^2 = R^2 + 2zf + f^2 \end{aligned} \tag{33}$$

and

$$\xi = \frac{r_1 + r_2}{2f}, \quad \eta = \frac{r_1 - r_2}{2f} \tag{34}$$

Hence for $R \gg f$,

$$\xi \approx \frac{R}{f}\left[1 + \frac{zf}{R^2} + \frac{f^2}{2R^2} + \ldots\right], \quad \eta = \frac{z}{R}\left[1 - \left(\frac{f}{R}\right)^3 + \ldots\right] \tag{35}$$

Substituting into equation (32), V_o simplifies to, for large R,

$$V_o = \frac{q}{\epsilon_o R} - Ez + \frac{Ef^3\xi_o}{3Q_1(\xi_o)}\cdot\frac{z}{R^3} \tag{36}$$

The first term on the right hand side is the Coulomb potential due to a point charge q in a medium of dielectric constant ϵ_o, the second term is due to the applied electric field and the third term to an electric dipole aligned in the z-direction. $Q_1(\xi_o)$, which was defined in equation (8), may be expressed in terms of the eccentricity $e = 1/\xi_o$ (equation (2)) as

$$Q_1(\xi_o) = \frac{1}{2e}\ln\left[\frac{1+e}{1-e}\right] - 1 \tag{37}$$

Using equation (30) for the volume of the ellipsoid, it is then readily verified that the moment m of the dipole (in a medium of dielectric constant ϵ_o the potential is $mz/\epsilon_o R^3$) is

$$m = \frac{\epsilon_o Ef^3\xi_o}{3Q_1(\xi_o)} = \frac{\epsilon_o E\Omega}{4\pi n(z)} \tag{38}$$

where the depolarization coefficient

$$n(z) = \frac{(1-e^2)}{2e^3}\left[\ln\left(\frac{1+e}{1-e}\right) - 2e\right] \tag{39}$$

depends only on the shape of the ellipsoid. This result is given (for ϵ_o = 1) by Landau and Lifshitz [23].

Electrostatic Free Energy of a Charged Conducting Ellipsoid

We adapt a method proposed by Landau and Lifshitz [23] for finding the electrostatic free energy associated with a charged ellisoid. Let the major axis of the ellisoid be along the z-axis as before. In place of the uniform electric field E parallel to this axis, we imagine a charge Q situated at a great distance D from the ellipsoid on the negative side of the z-axis. The uniform field E has therefore been replaced by a field of magnitude $Q/\epsilon_o D^2$ which, in the immediate vicinity of the ellipsoid, is almost uniform and points in the positive z-direction. It is convenient to regard the charge Q as located on a spherical conductor of radius r which is very much smaller than D, of course. Also, since D is very large, Q will be large to give a non-zero E. As before, the charge on the ellipsoid equals q.

The electrostatic free energy associated with the charged conducting ellipsoid and the charge Q equals

$$\frac{\epsilon_o}{8\pi} \int_{\vartheta} |\text{grad } V_o|^2 dv \qquad (40)$$

Here, the two conductors at separation D are embedded in an even larger volume of the oil medium, ϑ, which we shall allow to become infinite. Applying a form of Green's theorem and noting that $\nabla^2 V_o = 0$, we transform from a volume integral to surface integrals, namely

$$\frac{\epsilon_o}{8\pi}\int_{\vartheta}|\text{grad } V_o|^2 dv = \frac{\epsilon_o}{8\pi}\int_{S_1+S_2+S_3} V_o \frac{\partial V_o}{\partial n}\, dS \tag{41}$$

where S_1 is the surface of the perfectly conducting spheroid, S_2 that of the spherical conductor of radius r, and S_3 is the outer boundary of the dielectric (oil) medium. $\partial/\partial n$ denotes differentiation outwards from the volume ϑ normal to the surfaces. If S_3 is a sphere of radius D' where D' >> D, then $V_o(\partial V_o/\partial n)$ is proportional to $1/D'^3$ on S_3 and hence the integral over S_3 tends to zero as $D' \to \infty$. Let $V_o = \Phi$ on the ellipsoidal conductor, $\Phi_s = Q/\epsilon_o r$, the self-potential of the conducting sphere of radius r^3, and Φ_r be the residual potential at the conducting sphere of radius r due to the presence of the ellipsoid. Then we write

$$\frac{\epsilon_o}{8\pi}\int_{S_1+S_2+S_3} V_o \frac{\partial V_o}{\partial n}\, dS = \frac{q}{2}\Phi + \frac{Q}{2}(\Phi_s+\Phi_r) = \frac{q}{2}\Phi + \frac{Q^2}{2\epsilon_o r} + \frac{Q}{2}\Phi_r \tag{42}$$

by applying Gauss's theorem. A dipole moment m pointing in the positive direction is induced on the ellipsoidal conductor by the charge Q at D (Figure 8). Thus the residual potential Φ_r may be equated to

$$\Phi_r = \frac{q}{\epsilon_o D} - \frac{m}{\epsilon_o D^2} \tag{43}$$

Noting that in the immediate vicinity of the ellipsoid the electric field due to Q is virtually uniform, we substitute equation (38) for m, identifying E with $Q/\epsilon_o D^2$. Thus we get

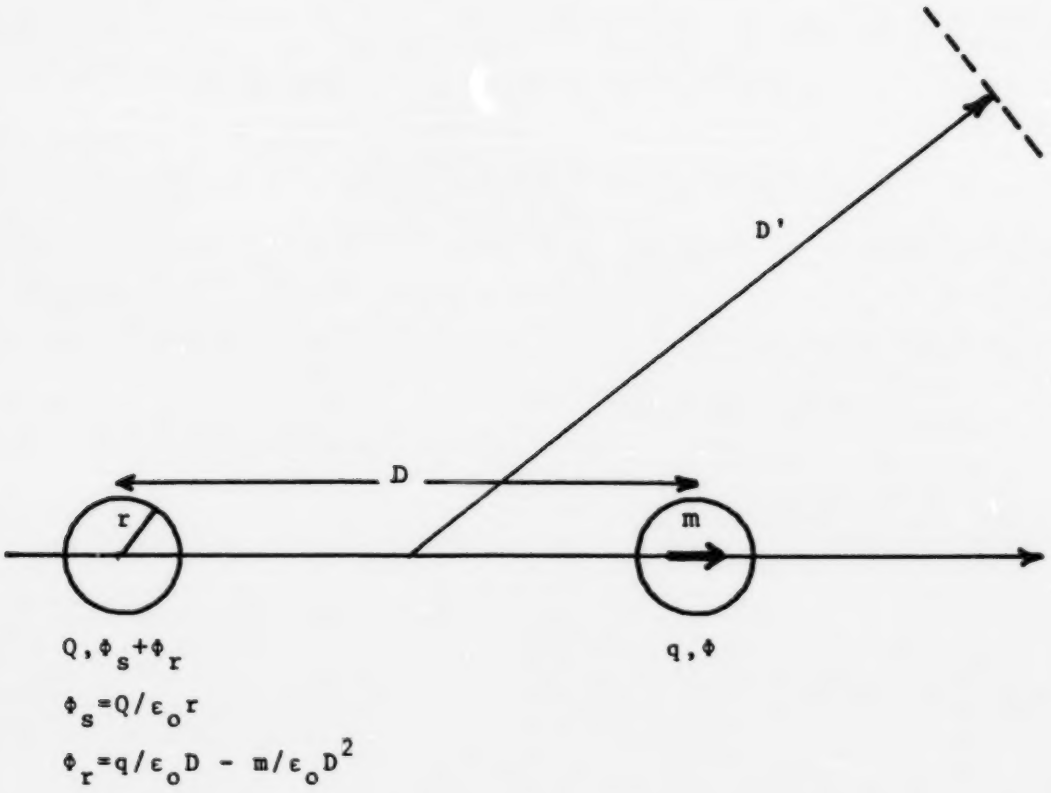

Figure 8. Illustration of Landau-Lifshitz type model for polarization energy of an ellipsoid in a uniform electric field.

$$\frac{Q}{2}\Phi_r - \frac{Qq}{2\epsilon_o D} - \frac{Q\mu}{2\epsilon_o D^2} - \frac{q}{2}DE - \frac{QE\Omega}{8\pi n(z)}$$

$$- \frac{q}{2}DE - \frac{\epsilon_o E^2 \Omega}{8\pi n(z)} \qquad (44)$$

It would appear that we need to subtract away two terms, both of which become infinite when D tends to infinity. One is the self-energy term $Q^2/(2\epsilon_o r)$ $(= E^2\epsilon_o D^4/2r)$ in equation (42) and the other is qDE/2 in equation (44). The omission of the self-energy term seems physically justified, but the subtraction of qDE/2 is not obvious. In the analysis of Landau and Lifshitz [23], a problem of this nature did not arise since they put $q = 0$. Two distances D' and D, both tending to infinity, occur in our evaluation of the electrostatic energy in equation (41). Firstly, keeping D fixed, we allow D' to tend to infinity, thereby eliminating the integral over S_3 in equation (41). On imagining D to become infinite, it seems that we need to subtract away qDE/2 as well as a self-energy term. Some independent way of arriving at the induced dipole energy term in equation (44) is desirable.

The required electrostatic free energy associated with the conducting charged ellipsoid under the influence of the applied electric field will be assumed to be

$$F_q + F_d = \frac{q}{2}\Phi - \frac{\epsilon_o E^2 \Omega}{8\pi n(z)} \qquad (45)$$

The first term is the classical expression for the energy of a conductor at potential Φ and carrying a total charge q. We assume that the electric field due to the charge Q at distance D is equivalent to a uniform electric

field E and hence can apply the results obtained above for the potential of a perfectly conducting ellipsoid under an electric field. From equations (6) and (26) and the relation $\xi_0 = 1/e$ (see equation (2)), we can write

$$F_q = \frac{q\Phi}{2} = \frac{q^2}{2\epsilon_o f} Q_o(\xi_o) = \frac{q^2}{4\epsilon_o f} \ln\left(\frac{1+e}{1-e}\right) \tag{46}$$

The radius s of a sphere having the same volume as the ellipsoid is given by $s^3 = ab^2$. It follows from equations (2) and (30) that

$$F_q = \frac{q\Phi}{2} = \frac{q^2}{4\epsilon_o s e}(1-e^2)^{1/3} \ln\left(\frac{1+e}{1-e}\right) \tag{47}$$

Panchenkov and Tsabek [29] derive an expression for the energy of the "intrinsic-charge" field which differs from equation (47) in that their dielectric constant is in the numerator, but this must be an error. On putting $\Omega = 4\pi s^3/3$ where s is the radius of the sphere with the same volume as the ellipsoid, the second term in equation (45) becomes

$$F_d = \frac{-\epsilon_o E^2 \Omega}{8\pi n^{(z)}} = \frac{-\epsilon_o E^2 s^3}{6n^{(z)}} \tag{48}$$

which agrees with the expression obtained by Panchenkov and Tsabek [29]. However, their derivation is far from clear. Indeed the integral from which they obtain their result equals zero. The parameter $n^{(z)}$ in equation (48) is given by equation (39).

Determination of Eccentricity of Ellipsoid

We write the total free energy as the sum of the three energies due to surface tension F_s, to charge on the ellipsoidal conductor F_q, and to the induced dipole moment F_d, as given by equations (31), (47) and (48) respectively. We seek the minimum of

$$F = F_s + F_q + F_d \tag{49}$$

with respect to the eccentricity e at constant volume Ω or constant radius s of the equivalent sphere. The two conditions to be satisfied are

$$\frac{\partial F}{\partial e} = 0, \quad \frac{\partial^2 F}{\partial e^2} > 0 \tag{50}$$

From the three equations referred to above

$$\frac{\partial F_s}{\partial e} = \frac{2\pi\gamma s^2}{3e}\left[\frac{(3-2e^2)}{(1-e^2)^{2/3}} - (3-4e^2)\frac{\sin^{-1}e}{e(1-e^2)^{7/6}}\right]$$

$$\approx \frac{32}{45}\pi\gamma s^2 e^3\left(1 + \frac{43}{42}e^2\right) \tag{51}$$

$$\frac{\partial F_q}{\partial e} = \frac{-q^2}{12\epsilon_o s}\frac{\left[(3-e^2)\ln\left(\frac{1+e}{1-e}\right) - 6e\right]}{e^2(1-e^2)^{2/3}} \approx \frac{-2}{45}\cdot\frac{q^2e^3}{\epsilon_o s}\left(1 + \frac{32}{21}e^2\right) \tag{52}$$

$$\frac{\partial F_d}{\partial e} = \frac{\epsilon_o E^2 s^3}{6}\frac{\left[\frac{1}{e} + \left(\frac{1}{2e^2} - \frac{3}{2e^4}\right)\left[\ln\left(\frac{1+e}{1-e}\right) - 2e\right]\right]}{\left[\frac{(1-e^2)}{2e^3}\left[\ln\left(\frac{1+e}{1-e}\right) - 2e\right]\right]^2} \approx \frac{-2}{5}\epsilon_o E^2 s^3 e\left(1 + \frac{6}{7}e^2\right) \tag{53}$$

where we have put $\Omega = 4\pi s^3/3$. We have expanded in powers of e and retained the leading term, assuming e to be small compared with 1. For small e, the first condition in equation (50) yields

$$e^2 = \frac{q\epsilon_o E^2 s^4}{16\pi\gamma s^3 - q^2/\epsilon_o} \tag{54}$$

a result which differs from that quoted by Panchenkov and Tsabek [29] since they write $\epsilon_0 q^2$ instead of q^2/ϵ_0. Thus they overestimate the effect of the charge on the conducting ellipsoid by the factor $\epsilon_0^2 \sim 10$. Taylor [2] quotes the relation (54) for an uncharged ellipsoid ($q = 0$). Clearly $16\pi\gamma s^3 > q^2/\epsilon_0$ for equation (54) to be meaningful. The second condition in equation (50) for small e, which corresponds to equation (54) is

$$\frac{32}{15}\pi\gamma s^2 e^2 - \frac{2q^2}{15\epsilon_o s}e^2 - \frac{2}{5}\epsilon_o E^2 s^3 > 0 \tag{55}$$

This is obviously satisfied when equation (54) is substituted for e^2.

To obtain some idea of the magnitude of the eccentricity e, assume thaat $q = 0$ and choose $E = 1$ kV/cm, $\gamma = 30$ dynes/cm = 30 mN/m, $\epsilon_0 = 3$ and $S = 100$ μm. Then equation (54) yields $e^2 = 1-(b/a)^2 = -.0179$, so that $e = 0.134$ and $b/a = 0.991$, which indicates very small departure from a spherical droplet. The effect of a charge q on the droplet is most readily seen by writing equation (54) as

$$e^2 = \frac{q\epsilon_o E^2 s}{16\pi\gamma}\Big/\left[1 - \frac{\pi s\sigma^2}{\gamma\epsilon_o}\right], \quad \sigma = \frac{q}{4\pi\sigma^2} \tag{56}$$

where σ is the surface charge per unit area on the droplet. Clearly the magnitude of σ must be appreciably less than $(\gamma\epsilon_o/\pi s)^{1/2}$ for equations (54) or (55) to apply. For the values of ϵ_0, γ, and S chosen, $|\sigma| < 53.5$ e.s.u. $= 0.178$ μcoulombs/cm^2. When the departure from a spherical shape becomes appreciable, the approximation of equation (54) cannot be used and the full equations (51) and (53) are required. An increase of e with q at constant volume of droplet is to be expected since the area of the droplet surface would increase.

APPENDIX E

Introduce cylindrical polar coordinates ρ, z, ϕ, where the x,y coordinates in equation (1) become $x = \rho\cos\phi$, $y = \rho\sin\phi$ so that $\rho = f[(\xi^2-1)(1-\eta^2]^{1/2}$. The equation of the prolate ellipsoid is

$$\frac{z^2}{a^2} + \frac{\rho^2}{b^2} = 1 \tag{E1}$$

which also represents an ellipse in the Cartesian z-ρ plane. We can write

$$F_s = 4\pi\gamma\int_{\rho=0}^{\rho=b} \rho ds \tag{E2}$$

where ds is an element of length on the ellipse, i.e.

$$ds = (dz^2 + d\rho^2)^{1/2} = d\rho\left[1 + \left(\frac{dz}{d\rho}\right)^2\right]^{1/2} = d\rho\left[1 + \frac{\rho^2 a^2}{b^2(b^2-\rho^2)}\right]^{1/2} \tag{E3}$$

and we integrate in equation (E2) over a quarter of the ellipse from $\rho = 0$ to $\rho = b$. The surface tension energy becomes, on substituting $\rho = b\sin\theta$,

$$4\pi\gamma\int_0^b \left[1 + \frac{\rho^2 a^2}{b^2(\rho^2-a^2)}\right]^{1/2} d\rho = 4\pi\gamma b\int_0^{\pi/2} \sin\theta[b^2\cos^2\theta + a^2\sin^2\theta]^{1/2}\, d\theta \tag{E4}$$

From equation (2), the eccentricity of the ellipsoid is

$$e = (a^2-b^2)^{1/2}/a \tag{E5}$$

Substituting $u = \cos\phi$, equation (E4) yields for the surface tension energy

$$F_s = 2\pi\gamma b\int_0^1 [a^2 - (a^2-b^2)u^2]^{1/2}\, du = 4\pi\gamma bae\int_0^1 \left[\frac{1}{e^2} - u^2\right] du \qquad (E6)$$

The result of integration is equation (29).

VII. DISCUSSION AND CONCLUSIONS

In our calculation of the electrostatic interaction of the two conducting spheres in the presence of an applied uniform electric field E, we have divided the energy into two parts. The difference between equations (III.45) and (III.46) yields that part of the interaction which is due to the charges Q_1 and Q_2 on the two spheres. This difference is obtained as a quadratic expression in Q_1 and Q_2, which also contains linear terms in Q_1 and Q_2. These linear terms are proportional to $E\cos\psi$, where ψ is the angle between the line of centres of the two spheres and the direction of the applied field. The parameters λ_1, τ_2, λ_2, τ_3, and λ_3 which appear in the coefficients of the quadratic in Q_1 and Q_2 are functions of the bipolar coordinates μ_1 and μ_2 involving a number of infinite series. Given the radii R_1 and R_2 of the two spheres and the separation between the centres of the spheres R, the coordinates μ_1 and μ_2 can be obtained from the relations

$$R = D_1 + D_2 = c(\coth\mu_1 + \coth\mu_2) = R_1\cosh\mu_1 + R_2\cosh\mu_2 \qquad (1)$$

$$c = R_1\sinh\mu_1 = R_2\sinh\mu_2 \qquad (2)$$

Having determined μ_1 and μ_2, the parameter c is defined in equation (2).

In the case of two uncharged spheres, the electrostatic energy due to the applied field is given in equation (IV.15), namely

$$-\tfrac{1}{2}E(m_x\sin\psi + m_z\cos\psi) \qquad (3)$$

where the components of the total dipole moment characterizing the two spheres m_x and m_z are given by equations (IV.35) and (IV.33) respectively. The coefficients A_n and B_n in equation (IV.33) are expressed in terms of μ_1, μ_2, E, and ψ in equations (IV.22) and (IV.23) respectively. The corresponding formulae for the coefficients C_n and D_n in equation (IV.35) are given in equations (IV.24) and (IV.25) respectively. Thus m_x and m_z are expressed in terms of μ_1, μ_2, E, and ψ. At infinite separation between the two spheres, m_x and m_z become

$$m_x^0 = \epsilon_o(R_1^3 + R_2^3)\sin\psi \ , \quad m_z^0 = \epsilon_o(R_1^3 + R_2^3)\cos\psi \tag{4}$$

It follows that the electrostatic interaction energy of the two uncharged spheres in the presence of the applied electric field is obtained by subtracting from equation (3) the energy

$$-\tfrac{1}{2}E\epsilon_o(R_1^3+R_2^3)(\cos^2\psi + \sin^2\psi) = -\tfrac{1}{2}E\epsilon_o(R_1^3+R_2^3) \tag{5}$$

The parameters that we can choose are the two radii of the spheres R_1 and R_2, the separation R, the two charges Q_1 and Q_2, the electric field E, and the angle ψ. Making use of equations (1) and (2), the electrostatic interaction energy is expressed in terms of μ_1, μ_2, E, and ψ, and is therefore readily computed in terms of R_1, R_2, R, E, and ψ using the various series derived in this report.

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